Universality & Control of Fat Tails

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Why control at the theoretical physics conference?

Why control at the theoretical physics conference?

- Think of it as a Modern Non-Equilibrium Stat Mech
- Applies, e.g., to Physical Systems (Lagrangian Swimmers)
- Landau Institute type of topics
  - Plenty of Universality (T exp)
  - Some Relation to Instantons

Outline

1. Multiplicative Noise
   - Multiplicative Noise
   - Fluid Mechanics: Swimmers
   - Thermal Control of Buildings
2. Control of Steady State
   - White-Gaussian-Multiplicative
   - Control of Swimmers
   - Thermal Control
3. Fat (Algebraic) Tails
   - Swimmers in "Batchelor" Flows
   - General Model:Synthesis
4. Stochastic Optimal Control
   - General Considerations
   - Swimmers: Stochastic Optimal Control
   - Thermal: Stochastic Optimal Control

Linear System Driven by Multiplicative Noise

$$\frac{dx_i}{dt} = \sum_j \left(m_{ij} + \sigma_{ij}(t)\right) x_j(t) + \xi_i(t) + u_i(t)$$

- $\boldsymbol{m} = (m_{ij} : \forall i,j = 1,\cdots,d)$=const
- $\boldsymbol{\sigma}(t) = (\sigma_{ij}(t) : \forall i,j)$ – zero-mean stochastic
- $\boldsymbol{\xi}(t) = (\xi_i(t) : \forall i)$ – zero-mean white-Gaussian
- $\boldsymbol{u}(t) = (u_i(t) : \forall i)$ – vector of control

$\boldsymbol{\sigma}(t)$ – Multiplicative Stochastic

- $\frac{d}{dt}\boldsymbol{W} = \boldsymbol{\sigma}\boldsymbol{W}$, $\boldsymbol{W}(t)$ – $T\exp$
- Oseledets theorem: at $t \to \infty$, $\log(\boldsymbol{W}^+\boldsymbol{W})/t \to$ const
  - $\boldsymbol{W}\boldsymbol{f}_i = c_i \boldsymbol{f}_i$, $\lambda_i = \log|\boldsymbol{W}\boldsymbol{f}_i|/t$, $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_d$
  - $P(\lambda_1,\cdots,\lambda_d|t) \propto \exp\left(-tS(\lambda_1,\cdots,\lambda_d)\right)$
  - $S(\cdots)$ – Crámer function

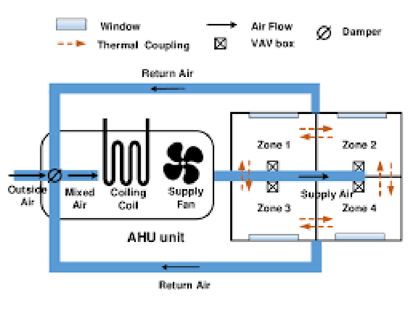
Active & Passive Swimmers

$$-\alpha\left(\frac{d\boldsymbol{r}}{dt}-\boldsymbol{\sigma}(t)\boldsymbol{r}\right)=\boldsymbol{u}(t)+\boldsymbol{\xi}(t)$$

- $\boldsymbol{u}$ – control exerted by an active swimmer:
  - keep in-sight
- $\boldsymbol{\sigma}(t)$ – fluctuating velocity gradient in “Batchelor" flow

Dynamics of Temperature in Multi-Zone Buildings

$$\frac{dT}{dt} = -c_o(T - T_o) - c_s(T - T_s)u(t) + \xi(t) \text{ [as seen from a zone]}$$



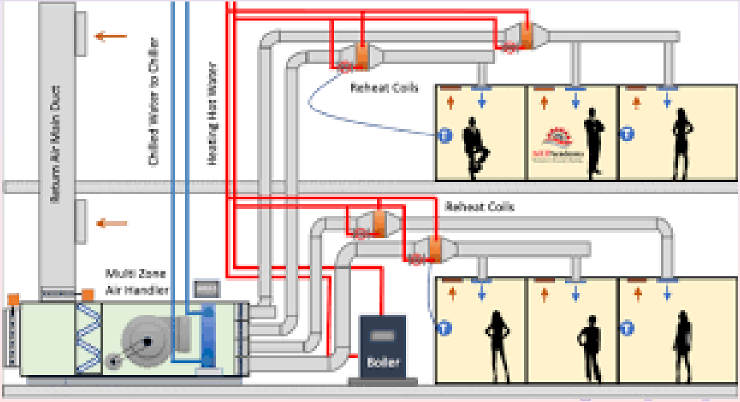
- T_o-outside and T_s-Air-Handling-Unit (AHU)

- $u(t)$ – control of the AHU opening
- Linearizing around "comfort" temperature/efforts
  - $0 = -\underline{c}_o(\underline{T} - T_o) - c_s(\underline{T} - T_s)\underline{u}$
  - $c_o = \underline{c}_o + \sigma(t)$
  - $u(t) = \underline{u} + \phi\theta$ (+ linear feedback)
  - $\theta = T - \underline{T}$
- $\frac{d\theta}{dt} = -c(\phi)\theta + \tilde{\xi}(t) - \sigma(t)\theta$
- $c(\phi) = c_0 + c_1\phi$, $c_0 = \underline{c}_o + c_s\underline{u}$, $c_1 = c_s(\underline{T} - T_s)$, $\tilde{\xi}(t) = \xi(t) + T_o\sigma(t)$

Dynamics of Temperature in Multi-Zone Buildings

Network (of zones) Generalization

- $\frac{d\theta_i}{dt} = -\left(c_i(\phi) + \sigma_{io}\right)\theta_i - \sum_{j\sim i}\left(c_{ij} + \sigma_{ij}\right)\left(\theta_i - \theta_j\right) + \xi_i(t)$



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White-Gaussian-Multiplicative: Fokker-Planck

- Multiplicative Noise = White Gaussian
- State Feedback Control: $\boldsymbol{u}(t) \to \boldsymbol{w}(\boldsymbol{x}(t))$ [prescribed]
- $\Rightarrow$ Fokker-Planck:
$$\left(\partial_{x_i}\left(w_i(\boldsymbol{x}) + m_{ij}x_j\right) + \kappa_{ij}\partial_{x_i}\partial_{x_j} + D_{ik;jl}\partial_{x_i}x_k\partial_{x_j}x_l\right) P(\boldsymbol{x}|\boldsymbol{w}) = 0$$

Steady State Control

$$\phi^* = \arg\min_{\phi} \bar{C}(\phi), \quad \bar{C}(\phi) = \int d\boldsymbol{x} P(\boldsymbol{x}|\boldsymbol{w}_\phi) C(\boldsymbol{x}, \boldsymbol{w}_\phi)$$

$$C(\boldsymbol{x}, \boldsymbol{w}_\phi) = \underbrace{C_c(\boldsymbol{w}_\phi)}_{\text{cost of control}} + \underbrace{C_g(\boldsymbol{x})}_{\text{cost of achieving the goal, e.g.}(\boldsymbol{x}\boldsymbol{x}^T)^{q/2}}$$

Consider Examples ...

Control of Swimmers

$$\boldsymbol{u}(t) \to \boldsymbol{w}_\phi(\boldsymbol{r}) = \phi \boldsymbol{r},\ \ C_c\{\boldsymbol{w}\} \to \boldsymbol{w}^2,\ \ C_g(\boldsymbol{r}) \to \beta r^q$$

Batchelor-Kraichnan model

- $\forall i,j,k,l:\ \mathbb{E}\left[\sigma_{ij}(t)\sigma_{kl}(t')\right] = D(d+1)\delta(t-t')\left(\delta_{jl}\delta_{ik} - \frac{\delta_{ij}\delta_{kl}+\delta_{jk}\delta_{il}}{d+1}\right)$

Fokker-Planck

- $r^{1-d}\frac{d}{dr}r^d\left(\phi + \frac{1}{2}\left(D(d-1)r + \frac{\kappa}{r}\right)\frac{d}{dr}\right)P(r|\phi) = 0$

Optimal Solution

- $P(r|\phi) \propto \left(\frac{\kappa}{D} + (d-1)r^2\right)^{-\phi/((d-1)D)}$
- "valid" at $\phi > (d-1)dD/2$
- optimal: $\phi^{(*)} = \frac{D(d+2)(d-1)+\sqrt{4\beta+D^2(d+2)^2(d-1)^2}}{2}$

Thermal Control (single zone)

Linear feedback, M-noise short correlated

Fokker-Planck → solution → optimal

- $\left(\partial_\theta c(\phi)\theta + \kappa\partial_\theta^2 + D(\partial_\theta\theta)^2\right) P(\theta|\phi) = 0$
- $P(\theta|\phi) = \sqrt{\frac{D}{\pi\kappa}} \frac{\Gamma\left(\frac{c(\phi)}{2D}+\frac{1}{2}\right)}{\Gamma\left(\frac{c(\phi)}{2D}\right)} \left(1+\frac{D\theta^2}{\kappa}\right)^{-\frac{1}{2}-\frac{c(\phi)}{2D}}$
- MqP-stable at $\phi > \phi^{(s)} = (D(\max(q,2)-1)-c_0)/c_1$.
- optimal: $\phi^* = \frac{2D-c_0+\sqrt{(2D-c_0)^2+\beta c_1^2}}{c_1}$

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Swimmers in Large Scale Flows

- Implicit Solution:
  $\boldsymbol{r}(t) = e^{-\phi t}\boldsymbol{W}(t)\left(\boldsymbol{r}(0) + \int_0^t dt' e^{\phi t'}\boldsymbol{W}^{-1}(t')\boldsymbol{\xi}(t')\right)$
- Largest Lyapunov exponent: $\lambda_1 = \max_i \lambda_i$
- Explicit Asymptotic: $r(t) \approx exp\left((\lambda_1(t) - \phi)t\right) r_d \gg r_d = \sqrt{|\lambda|/\kappa}$
- Large t, Cramér function: $P(\lambda_1|t) \propto \exp\left(-tS_1(\lambda_1)\right)$
- From statistics of λ_1 to statistics of r: $P(r|t)\frac{r^{d-1}}{r_d^d} \to$
  $$\exp\left(-t\left(S_1(\bar{\lambda}_1) + \left(\frac{1}{t}\log\left(\frac{r}{r_d}\right) + \phi - \bar{\lambda}_1\right)^2 S_1''(\bar{\lambda}_1)\right)\right)$$
  $$\to \Big|_{t\to\infty;\ r\gg r_d} P_{st}(r)\frac{r^{d-1}}{r_d^d} \propto \left(\frac{r_d}{r}\right)^{2(\phi-\bar{\lambda}_1)S_1''(\bar{\lambda}_1)}$$
- Stationary if: $\phi > \bar{\lambda}_1 + 1/(2S''(\bar{\lambda}_1))$

General Model: Synthesis

- Linear feedback: $u_i(t) \to \sum_j \phi_{ij} x_j$
- $\boldsymbol{x}(t) = \exp(-(\boldsymbol{m}+\phi)t)\boldsymbol{W}(t)\tilde{x}$,
- $\tilde{x}$ stabilizes to a constant as t grows
- $\log P_{st}(\boldsymbol{x}\boldsymbol{f}_i^T) \propto 2\boldsymbol{f}_i\,(\boldsymbol{m}+\phi)\,\boldsymbol{f}_i^T S_i''(0) \log \frac{x_d}{\boldsymbol{x}\boldsymbol{f}_i^T}$
- Dependence on $\tilde{x}$ is "under logarithm" – thus weak and replaced by x_d
- Statistics of any norm of $\boldsymbol{x}$ is equivalent to statistics of $(\boldsymbol{x}\boldsymbol{f}_1^T)$ associated with the largest Lyapunov exponent

use white- multiplicative noise

- when control is slower than $1/\bar{\lambda}_1$

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General Considerations

Hamilton-Jacobi-Bellman (HJB)

- finite time horizon
- white – effectively short-correlated – multiplicative noise
- cost-to-go (action): $S(t,\boldsymbol{x})$ – function of t and $\boldsymbol{x}(t)$:
  $S(t,\boldsymbol{x}) = \min_{\{\boldsymbol{u}(t)\}}\Big(S_f(\boldsymbol{x}(t_f)) + \int_t^{t_f} d\tau\, \mathbb{E}\left[C(\boldsymbol{x}(\tau),\boldsymbol{u}(\tau))\right]\Big),\ S(t_f,\boldsymbol{x}) = S_f(\boldsymbol{x})$
- HJB:
  $-\partial_t S(t,\boldsymbol{x}) = \min_{\boldsymbol{u}}\Big(C(\boldsymbol{x},\boldsymbol{u}) + \left(\underline{a}_i + u_i + \underline{m}_{ij}x_j\right)\partial_{x_i}S(t,\boldsymbol{x}) + \left(\kappa_{ij}\partial_{x_i}\partial_{x_j} + D_{ik;jl}x_k\partial_{x_i}x_l\partial_{x_j}\right)S(t,\boldsymbol{x})\Big)$

Swimmers: Stochastic Optimal Control

- $\boldsymbol{u}$ is elongated with $\boldsymbol{r}$: $\boldsymbol{u} = u\boldsymbol{r}/r \;\Rightarrow\; C(u,r) = u^2 + \beta r^q$
- $-\partial_t S = \beta r^q + \frac{r^{1-d}}{2}\partial_r r^{d-1}\left(D(d-1)r^2 + \kappa\right)\partial_r S - \frac{1}{4}\left(\partial_r S\right)^2$
- optimal control: $u^*(t,r) = -\partial_r S(t,r)/2$
- at $q=2$ $S_f(r)$ is quadratic in $r \Rightarrow S(t,r) = \varsigma(t) r^2 + s(t) \Rightarrow$ $d\kappa\varsigma + ds/dt = 0,\; (d^2+d-2)D\varsigma - \varsigma^2 + \beta + d\varsigma/dt = 0$
- $\Rightarrow \varsigma(t) = \frac{1}{2}\Bigg(D(d+2)(d-1) + \sqrt{4\beta + D^2(d+2)^2(d-1)^2}$ $\times \tanh\left(\frac{(t_1-t)}{2}\sqrt{4\beta + D^2(d+2)^2(d-1)^2}\right)\Bigg)$ where t_1 is tuned to satisfy, $S(t_f, r) = S_f(r)$

Thermal: Stochastic Optimal Control

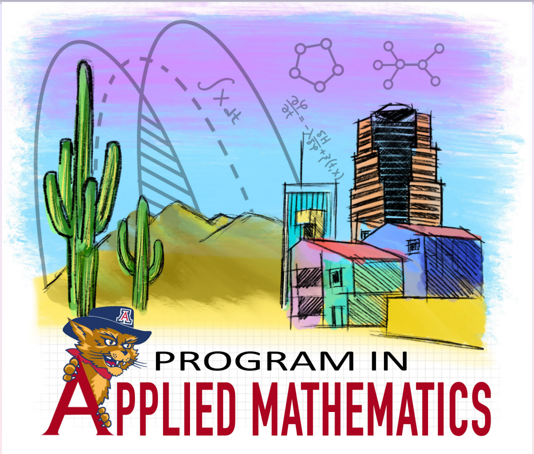
Multi-zone, short-correlated

- Hamilton-Jacobi-Bellman: $-\partial_t S =$
$$\sum_{i\in\mathcal{V}}\left(\beta_i|\theta_i|^q + D_{io}(\theta_i\partial_{\theta_i})^2 S + \kappa_i\partial_{\theta_i}^2 S - \frac{c_{is}^2(\underline{T}-T_s)^2}{4\alpha_i}\left(\partial_{\theta_i}S\right)^2\right) +$$
$$\sum_{\{i,j\}\in\mathcal{E}}(\theta_i-\theta_j)(\partial_{\theta_i}-\partial_{\theta_j})\left(\underline{c}_{ij} + D_{ij}(\theta_i-\theta_j)(\partial_{\theta_i}-\partial_{\theta_j})\right)S$$
- Linear-Quadratic-Gaussian: $S(t,\boldsymbol{\theta}) = \sum_{i,j}\theta_i\varsigma_{ij}(t)\theta_j + s(t)$
  - $\Rightarrow$ system of generalized Riccati equations

Conclusions & Path Forward

- Analyzed linear dynamic system driven by additive and multiplicative noise, stabilized by feedback
- $\Rightarrow$ algebraic tail (when stabilized)
- Explicit expression for the tail' exponent
- Examples from Fluid Mechanics (FM) and Civil Engineering (CE)

- Extend to complex cases – polymer solutions, multi-zone engineered systems
- Towards data driven approaches,
  e.g. via reinforcement learning



Support is Appreciated !!

- UArizona Funds

- IEEE Control Syst. Lett. 2023
- Control & Decision Conference 2023
- arXiv:2303.09635

Thanks for your attention !