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Generalized multifractality at 2D Anderson transitions

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J.F. Karcher, N. Charles, I.A. Gruzberg, A.D.M., Ann.Phys. 435, 168584 (2021)

J.F. Karcher, I.A. Gruzberg, A.D.M., PRB 105, 184205 (2022)

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J.F. Karcher, I.A. Gruzberg, A.D.M., PRB 107, 104202 (2023)

S.S. Babkin, J.F. Karcher, I.S. Burmistrov, A.D.M., arXiv (2023)

J. Karcher, **Karlsruhe → Penn State**

I. Gruzberg, **Ohio**

N. Charles, Ohio

S. Babkin, IST Austria

I. Burmistrov, Landau Inst.

related preceding papers:

I.A.Gruzberg, A.W.W.Ludwig, A.D.M., M.R.Zirnbauer, PRL 107, 086403 (2011)

I.A. Gruzberg, A.D.M., M.R. Zirnbauer, PRB 87, 125144 (2013)

Anderson localization

Philip W. Anderson

1958 "Absence of diffusion in certain random lattices"

Quantum particle moving in a random potential

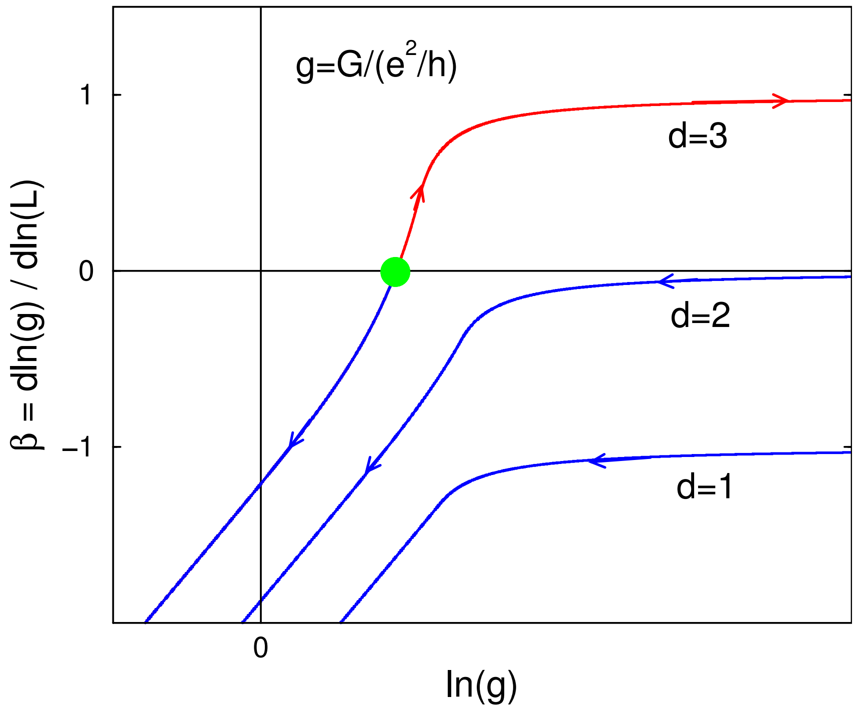
sufficiently strong disorder $\longrightarrow$ quantum localization

$\longrightarrow$ eigenstates exponentially localized, no diffusion

$\longrightarrow$ Anderson insulator

Nobel Prize 1977

Anderson Metal-Insulator Transitions



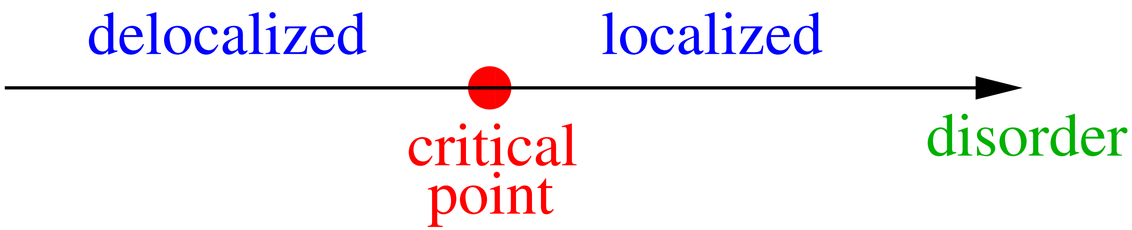
Connection with scaling theory of critical phenomena: Thouless '74; Wegner '76

Scaling theory of localization:
Abrahams, Anderson, Licciardello, Ramakrishnan '79

scaling variable:
dimensionless conductance $g = G/(e^2/h)$

RG for field theory: non-linear σ-model
Wegner '79

$d > 2$: **Anderson metal-insulator transition**



in some symmetry classes there is an Anderson MIT also in 2D

review: Evers, ADM, Rev. Mod. Phys. 80, 1355 (2008)

Field theory: non-linear σ-model

action:

$$S[Q] = \frac{\pi\nu}{4} \int d^d\mathrm{r}\ \mathrm{Tr}\,[-D(\nabla Q)^2 - 2i\omega\Lambda Q], \qquad Q^2(\mathrm{r}) = 1$$

Wegner'79

σ-model manifold: symmetric space

e.g., unitary Wigner-Dyson symmetry class:

- bosonic replicas: $\mathcal{M}_B = \mathrm{U}(n,n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $\quad n \to 0$

non-compact ("hyperboloid")

- fermionic replicas: $\mathcal{M}_F = \mathrm{U}(2n)/\mathrm{U}(n) \times \mathrm{U}(n)$, $\quad n \to 0$

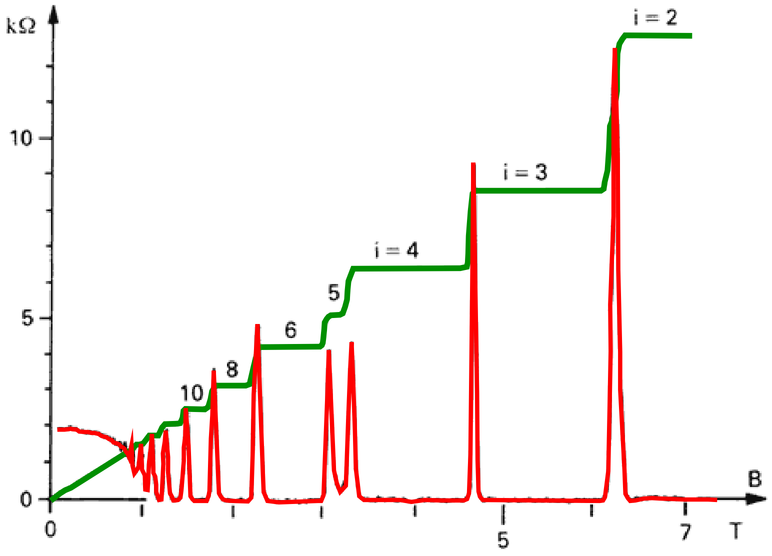
compact ("sphere")

- supersymmetry (Efetov'83): $\mathcal{M} = \mathrm{U}(1,1|2)/\mathrm{U}(1|1) \times \mathrm{U}(1|1)$

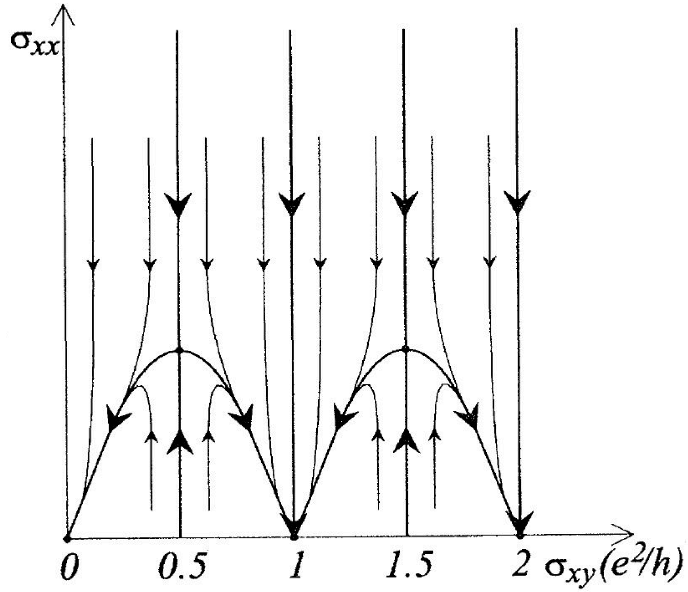
$\mathcal{M} = \{\mathcal{M}_B \times \mathcal{M}_F\}$ "dressed" by anticommuting variables

Anderson localization and topology

Paradigmatic example: Integer Quantum Hall Effect

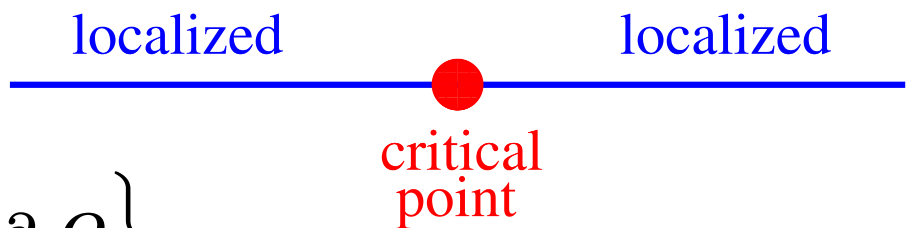


von Klitzing '80 ; Nobel Prize '85



IQHE flow diagram

Khmelnitskii' 83, Pruisken' 84



Field theory:

$\boldsymbol{\sigma}$-model with topological term

$$S = \int d^2r \left\{ -\frac{\sigma_{xx}}{8} \mathrm{Tr}(\partial_\mu Q)^2 + \frac{\sigma_{xy}}{8} \mathrm{Tr}\epsilon_{\mu\nu} Q \partial_\mu Q \partial_\nu Q \right\}$$

Anderson-localization transitions between topologically distinct insulating phases

Also: Anderson-localization critical theories emerge on surfaces of disordered topological insulators and superconductors.

Multifractality at Anderson transitions

$P_q = \int d^d r |\psi(\mathrm{r})|^{2\mathrm{q}}$ inverse participation ratio

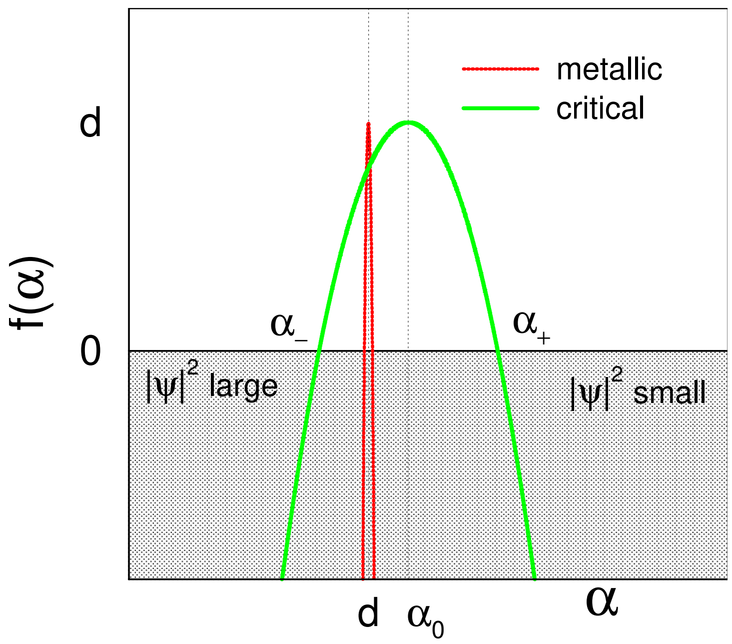
$$\langle P_q \rangle \sim \begin{cases} L^0 & \text{insulator} \\ L^{-\tau_q} & \text{critical} \\ L^{-d(q-1)} & \text{metal} \end{cases}$$

$\tau_q = d(q-1) + \Delta_q$ multifractality

normal anomalous

$\tau_q \longrightarrow$ Legendre transformation

$\longrightarrow$ singularity spectrum $f(\alpha)$



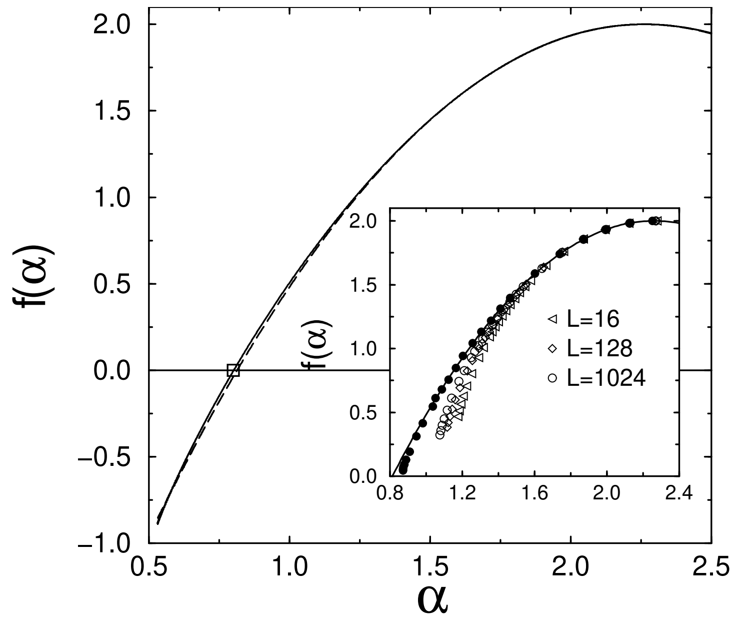
wave function statistics:

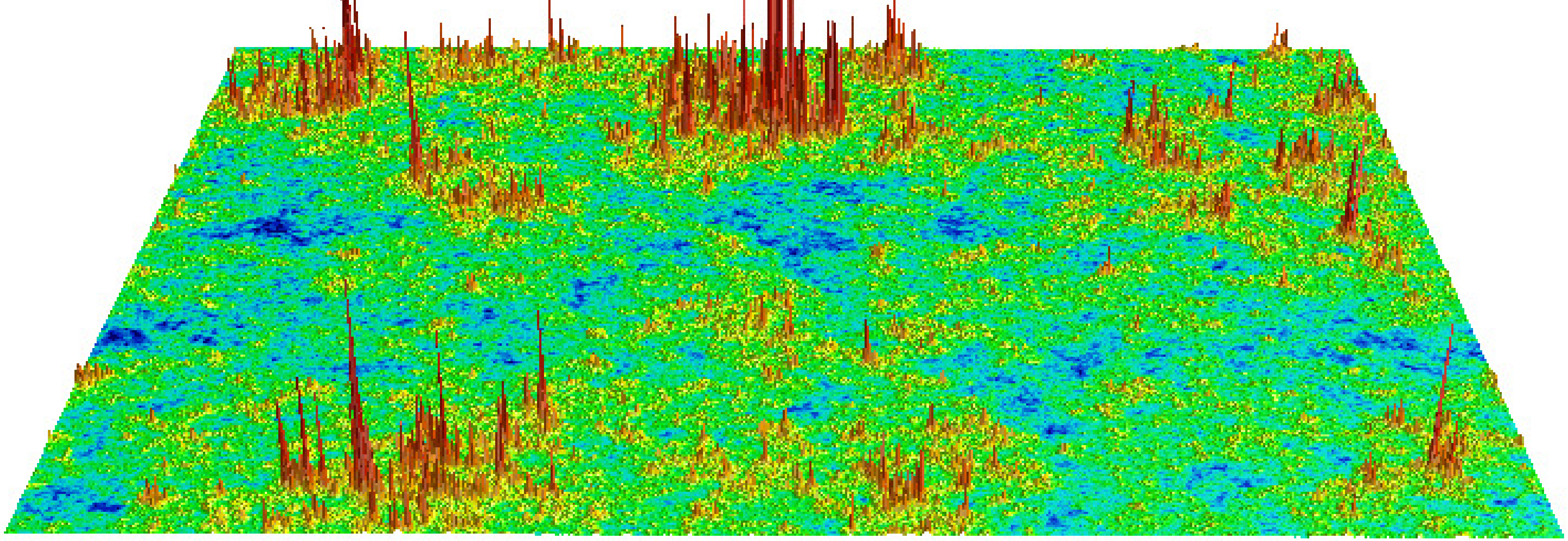
$\mathcal{P}(\ln |\psi^2|) \sim L^{-d+f(\ln |\psi^2|/\ln L)}$

$L^{f(\alpha)}$ – measure of the set of points where $|\psi|^2 \sim L^{-\alpha}$

Example: Multifractality at the Quantum Hall transition

Evers, Mildenberger, ADM '01





Multifractality in terms of local DOS

LDOS moments at criticality $\langle \rho^q \rangle \sim L^{-x_q}$

- **Wigner-Dyson classes:**

  average LDOS $\langle \rho \rangle$ is not critical

$$x_1 \equiv 0 \qquad x_q \equiv \Delta_q$$

- **Unconventional classes (chiral or particle-hole symmetry):**

  average LDOS in general critical $\langle \rho \rangle \sim L^{-x_1}$

$$\Delta_q = x_q - q x_1$$

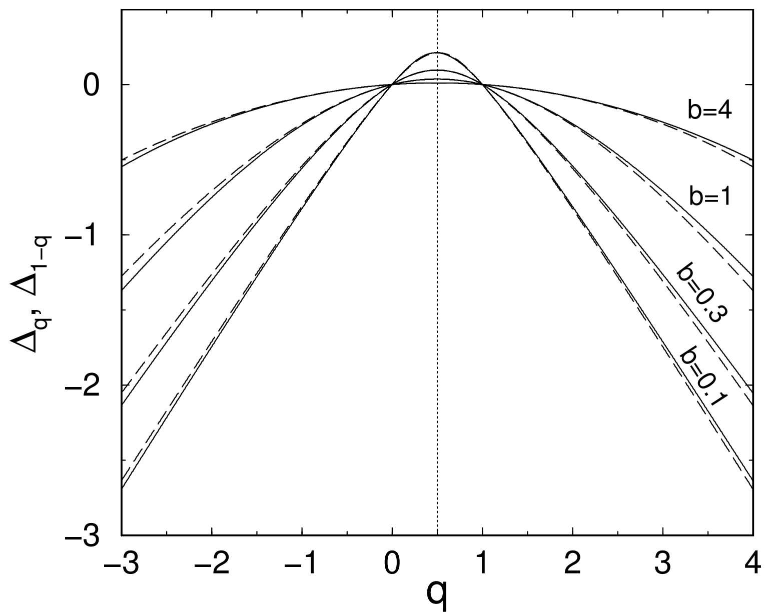
Symmetry of multifractal spectra (Wigner-Dyson classes)

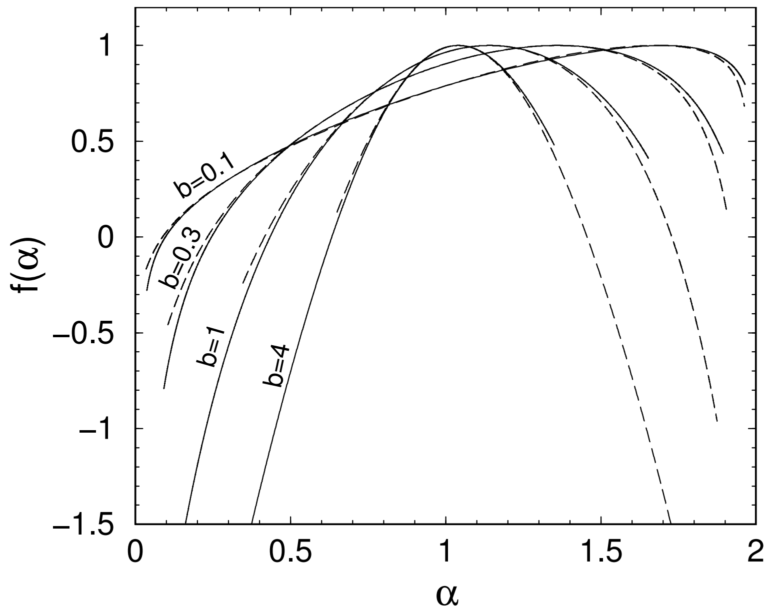
LDOS distribution in σ-model — ADM, Fyodorov '94; Fyodorov, Savin, Sommers '04-05

+ **universality** $\longrightarrow$ **exact symmetry of a multifractal spectrum:**

$$\Delta_q = \Delta_{1-q} \qquad\qquad f(2d-\alpha) = f(\alpha) + d - \alpha$$

ADM, Fyodorov, Mildenberger, Evers '06





$\longrightarrow$ **probabilities of unusually large and unusually small $|\psi^2(r)|$ are related !**

Disordered electronic systems: 10 symmetry classes

Wigner-Dyson classes

	$\hat{T}$	$\hat{P}$	$\hat{C}$	symbol
unitary				A
orthogonal	1			AI
symplectic	−1			AII

$\hat{T}$ – antiunitary, commutes with H,
$\hat{T}^2 = \pm 1$

$\hat{C}$ – unitary, anticommutes with H,
$\hat{C}^2 = 1$

$\hat{P}$ – antiunitary, anticommutes with H,
$\hat{P}^2 = \pm 1$

Chiral classes

	$\hat{T}$	$\hat{P}$	$\hat{C}$	symbol
unitary			1	AIII
orthogonal	1	1	1	BDI
symplectic	−1	−1	1	CII

$$H = \begin{pmatrix} 0 & \mathrm{t} \\ \mathrm{t}^\dagger & 0 \end{pmatrix}$$

Bogoliubov-de Gennes classes

$\hat{T}$	$\hat{P}$	$\hat{C}$	symbol
	−1		C
1	−1	1	CI
	1		D
−1	1	1	DIII

$$H = \begin{pmatrix} \mathrm{h} & \Delta \\ -\Delta^* & -\mathrm{h}^T \end{pmatrix}$$

Altland, Zirnbauer ’97

Disordered electronic systems: Sigma-model target spaces

Symmetry Class	NLσM (n-c|c)	Compact (fermionic) space	Non-compact (bosonic) space
A	AIII|AIII	$\mathrm{U}(2n)/\mathrm{U}(n)\times\mathrm{U}(n)$	$\mathrm{U}(n,n)/\mathrm{U}(n)\times\mathrm{U}(n)$
AI	BDI|CII	$\mathrm{Sp}(4n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$	$\mathrm{SO}(n,n)/\mathrm{SO}(n)\times\mathrm{SO}(n)$
AII	CII|BDI	$\mathrm{SO}(2n)/\mathrm{SO}(n)\times\mathrm{SO}(n)$	$\mathrm{Sp}(2n,2n)/\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)$
AIII	A|A	$\mathrm{U}(n)$	$\mathrm{GL}(n,\mathbb{C})/\mathrm{U}(n)$
BDI	AI|AII	$\mathrm{U}(2n)/\mathrm{Sp}(2n)$	$\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$
CII	AII|AI	$\mathrm{U}(n)/\mathrm{O}(n)$	$\mathrm{GL}(n,\mathbb{H})/\mathrm{Sp}(2n)$ $\equiv \mathrm{U}^*(2n)/\mathrm{Sp}(2n)$
C	DIII|CI	$\mathrm{Sp}(2n)/\mathrm{U}(n)$	$\mathrm{SO}^*(2n)/\mathrm{U}(n)$
CI	D|C	$\mathrm{Sp}(2n)$	$\mathrm{SO}(n,\mathbb{C})/\mathrm{SO}(n)$
BD	CI|DIII	$\mathrm{O}(2n)/\mathrm{U}(n)$	$\mathrm{Sp}(2n,\mathbb{R})/\mathrm{U}(n)$
DIII	C|D	$\mathrm{O}(n)$	$\mathrm{Sp}(2n,\mathbb{C})/\mathrm{Sp}(2n)$

Zirnbauer ’96

Generalized multifractality: Eigenfunction pure-scaling observables

$\lambda = (q_1, \ldots, q_n)$ labels representations.

Here q_j can be in general arbitrary complex numbers.

Building blocks: $\lambda = \underbrace{(1, \ldots, 1)}_{m \text{ times}} \equiv (1^m)$

"Spinless" symmetry classes: neither $\hat{T}^2 = -1$ nor $\hat{P}^2 = -1$

$\longrightarrow$ classes A, AI, AIII, BDI, D

$$P_{(1^m)}[\psi] = |\det(\psi_i(\mathbf{r}_j))_{m \times m}|^2 \equiv \left| \det \begin{pmatrix} \psi_1(\mathbf{r}_1) & \psi_2(\mathbf{r}_1) & \ldots & \psi_m(\mathbf{r}_1) \\ \psi_1(\mathbf{r}_2) & \psi_2(\mathbf{r}_2) & \ldots & \psi_m(\mathbf{r}_2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(\mathbf{r}_m) & \psi_2(\mathbf{r}_m) & \ldots & \psi_m(\mathbf{r}_m) \end{pmatrix} \right|^2$$

$\mathbf{r}_1, \ldots \mathbf{r}_\mathrm{m}$ — close spatial points, $\psi_1, \ldots \psi_m$ — close-in-energy eigenfunctions

For arbitrary $\lambda = (q_1, \ldots, q_n)$:

$$P_\lambda[\psi] = (P_{(1^1)}[\psi])^{q_1 - q_2} (P_{(1^2)}[\psi])^{q_2 - q_3} \cdots (P_{(1^{n-1})}[\psi])^{q_{n-1} - q_n} (P_{(1^n)}[\psi])^{q_n}$$

$\langle P_\lambda[\psi] \rangle$ are pure-scaling eigenfunction observables.

The proof goes via a mapping to the sigma model.

Generalized multifractality: Eigenfunction pure-scaling observables. Spinful case

"Spinful" symmetry classes: either $\hat{T}^2 = -1$ or $\hat{P}^2 = -1$ (or both)

$\longrightarrow$ Kramers-type (near-)degeneracy, classes **AII, CII, C, CI, DIII**

Building blocks: $\boldsymbol{\lambda} = (1^m)$

$$P_{(1^m)}[\psi] = \det \left(\begin{array}{c|c} (\psi_{i,\uparrow}(\mathbf{r}_j))_{m\times m} & (\psi_{\bar{\imath},\uparrow}(\mathbf{r}_j))_{m\times m} \\ \hline (\psi_{i,\downarrow}(\mathbf{r}_j))_{m\times m} & (\psi_{\bar{\imath},\downarrow}(\mathbf{r}_j))_{m\times m} \end{array} \right)$$

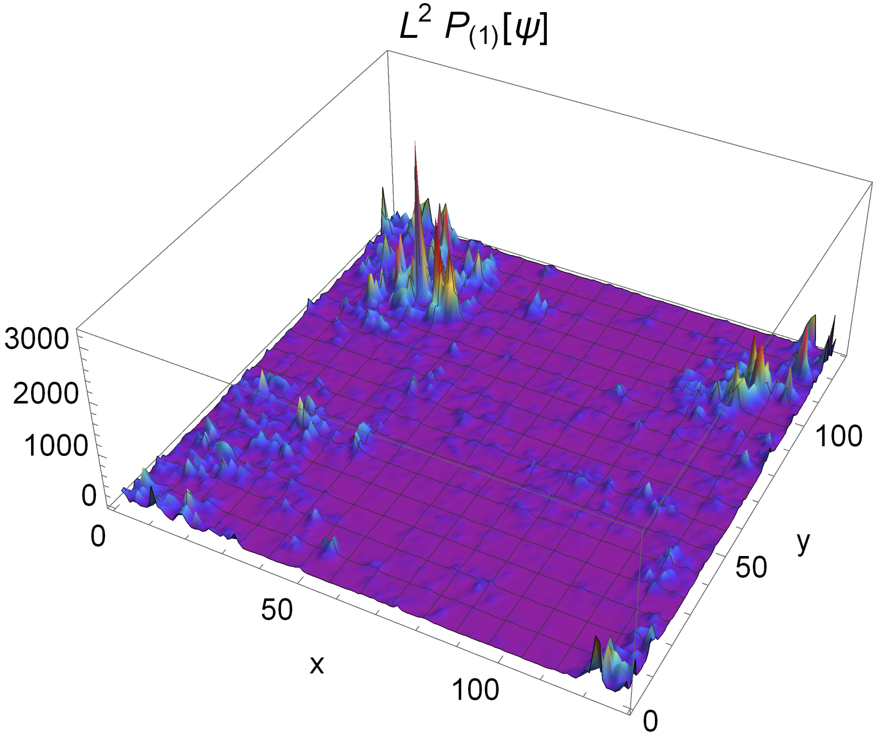
$$= \det \left(\begin{array}{ccc|ccc} \psi_{1,\uparrow}(\mathbf{r}_1) & \dots & \psi_{m,\uparrow}(\mathbf{r}_1) & \psi_{\bar{1},\uparrow}(\mathbf{r}_1) & \dots & \psi_{\bar{m},\uparrow}(\mathbf{r}_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \psi_{1,\uparrow}(\mathbf{r}_m) & \dots & \psi_{m,\uparrow}(\mathbf{r}_m) & \psi_{\bar{1},\uparrow}(\mathbf{r}_m) & \dots & \psi_{\bar{m},\uparrow}(\mathbf{r}_m) \\ \hline \psi_{1,\downarrow}(\mathbf{r}_1) & \dots & \psi_{m,\downarrow}(\mathbf{r}_1) & \psi_{\bar{1},\downarrow}(\mathbf{r}_1) & \dots & \psi_{\bar{m},\downarrow}(\mathbf{r}_1) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \psi_{1,\downarrow}(\mathbf{r}_m) & \dots & \psi_{m,\downarrow}(\mathbf{r}_m) & \psi_{\bar{1},\downarrow}(\mathbf{r}_m) & \dots & \psi_{\bar{m},\downarrow}(\mathbf{r}_m) \end{array} \right)$$

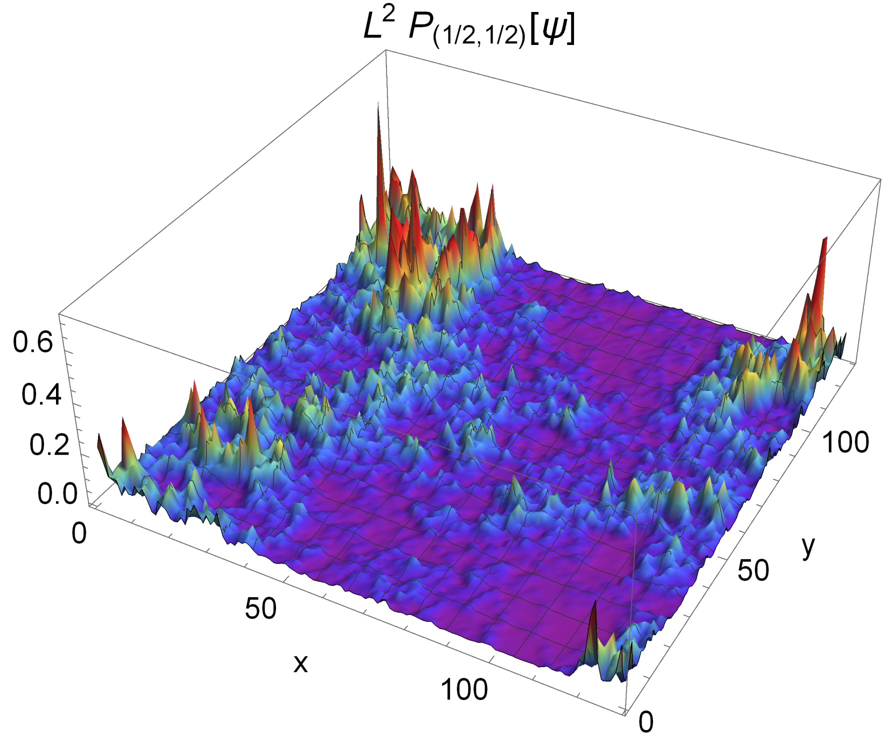
For arbitrary $\boldsymbol{\lambda} = (q_1, \dots, q_n)$:

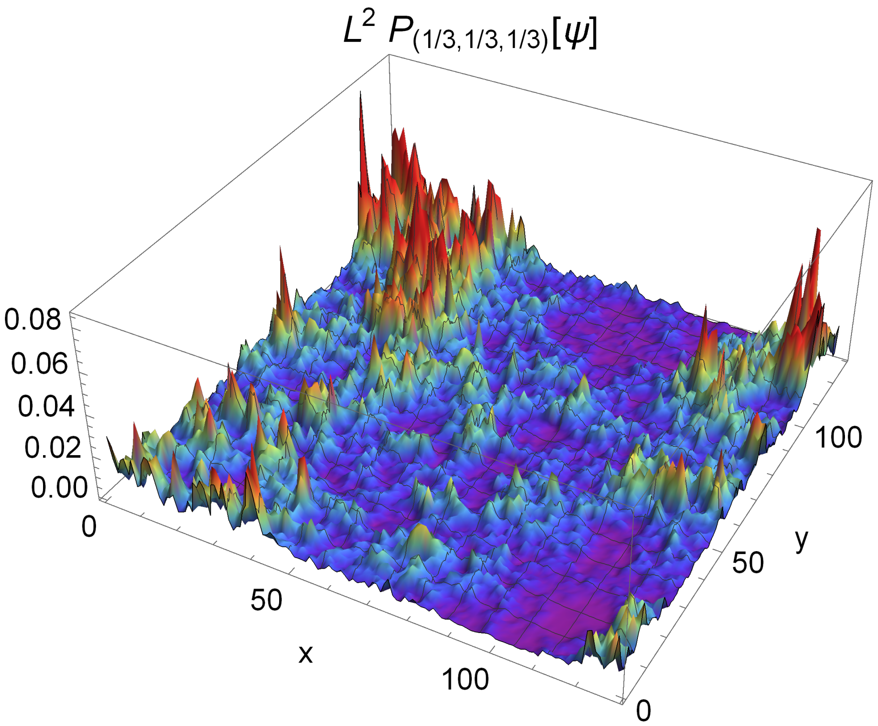
$$P_\lambda[\psi] = (P_{(1^1)}[\psi])^{q_1 - q_2} (P_{(1^2)}[\psi])^{q_2 - q_3} \cdots (P_{(1^{n-1})}[\psi])^{q_{n-1} - q_n} (P_{(1^n)}[\psi])^{q_n}$$

$\langle P_\lambda[\psi] \rangle$ **are pure-scaling eigenfunction observables.**

The proof goes via a **mapping to the sigma model.**







Example of spatial distribution of building blocks $L^2 P_{(1)}[\psi]$, $\quad L^2\left(P_{(1,1)}[\psi]\right)^{1/2}$, and $L^2\left(P_{(1,1,1)}[\psi]\right)^{1/3}$ for 2D metal-insulator transition of class AII

Scaling operators in sigma-model formalism

- Spinless classes: σ-model composite operators corresponding to wave function correlators $P_\lambda[\psi] \equiv P_{(q_1,\dots,q_n)}[\psi]$ are

$$\mathcal{P}_\lambda(Q) \equiv \mathcal{P}_{(q_1,\dots,q_n)}(Q) = d_1^{q_1-q_2} d_2^{q_2-q_3} \dots d_n^{q_n}$$

d_j — principal minor (determinant of a sub-block) of size $j \times j$ of the matrix

$$(1/2)(Q_{RR} - Q_{AA} + Q_{RA} - Q_{AR})_{bb}$$

- Spinful classes: Determinants $\longrightarrow$ Pfaffians of $2j \times 2j$ sub-blocks

These are pure scaling operators. Abelian fusion rules:

$$\mathcal{P}_\lambda(Q)\mathcal{P}_{\lambda'}(Q) = \mathcal{P}_{\lambda+\lambda'}(Q) \qquad P_\lambda[\psi]P_{\lambda'}[\psi] = P_{\lambda+\lambda'}[\psi]$$

A proof goes via Iwasawa decomposition $G = NAK$.

$\mathcal{P}_{(q_1,\dots,q_n)}(Q)$ are N-invariant spherical functions on G/K and have a form of “plane waves” on A.

Iwasawa decomposition and spherical functions

σ-model space: G/K $\quad$ K — maximal compact subgroup

- **Iwasawa decomposition: $G = NAK$ $\quad$ $g = nak$**

A — maximal abelian in G/K

N — nilpotent $\quad$ ($\longleftrightarrow$ triangular matrices with 1 on the diagonal)

Generalization of Gram-Schmidt (QR) decomposition:

matrix = triangular × unitary

- **Eigenfunctions of all G-invariant (Casimir) operators**

(in particular, RG transformation) are spherical functions on G/K.

- **N-invariant spherical functions on G/K are "plane waves"**

$$\varphi_{(q_1,\ldots,q_n)} = \exp\Big(-2\sum_{j=1}^{n} q_j x_j\Big)$$

$x_1, \ldots, x_n$ — natural coordinates on A.

- **$\phi_{(q_1,\ldots,q_n)}$ is exactly $\mathcal{P}_{(q_1,\ldots,q_n)}(Q)$ introduced above**

Weyl symmetries of scaling exponents

Weyl group acts in the space of weights λ (dual to Lie algebra of A) $\longrightarrow$ invariance of eigenvalues of any G invariant operator with respect to

(i) reflections: $q_j \to -c_j - q_j$

(ii) permutations: $q_i \to q_j + \frac{c_j - c_i}{2}; \qquad q_j \to q_i + \frac{c_i - c_j}{2}$

$c_j = 1 - 2j$,	class A	$c_j = -2j$,	class CI
$c_j = -j$,	class AI	$c_j = 2 - 2j$,	class DIII
$c_j = 3 - 4j$,	class AII	$c_j = 1 - 2j$,	class AIII
$c_j = 1 - 4j$,	class C	$c_j = 1/2 - j$,	class BDI
$c_j = 1 - j$,	class D	$c_j = 2 - 4j$,	class CII

$\longrightarrow$ **symmetries of eigenvalues of RG, i.e. scaling exponents** x_λ

$\longrightarrow$ **earlier found symmetry** $x_q = x_{1-q}$ **for Wigner-Dyson classes and many more symmetry relations for all classes**

Criticality in 2D

- **Broken spin-rotation invariance: Classes AII, D, DIII**

Metallic phase (weak antilocalization) and metal-insulator transition

- **Sublattice symmetry: Chiral classes AIII, BDI, CII**

Critical-metal phase and metal-insulator transition

- **Broken time-reversal invariance:**
Topological θ term and quantum Hall criticality

class A — Quantum Hall effect / transition
class C — Spin Quantum Hall effect / transition
class D — Thermal Quantum Hall effect / transition

- **Topologically protected criticality on surfaces of topological insulators and superconductors or in models of disordered Dirac fermions**

classes AII, CII — $\mathbb{Z}_2$ topological term
classes AIII, CI, DIII — Wess-Zumino term

Generalized multifractality — "fingerprint of a critical point"

Criticality in 2D

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Generalized multifractality — "fingerprint of a critical point"

Generalized multifractality and conformal invariance in 2D

- Usually (but not always) criticality implies conformal invariance
$\longrightarrow$ conformal field theory (CFT). The group of conformal transformations is particularly large in 2D (infinite-dimensional Virasoro algebra).

- Most of generalized multifractality correlation functions
depend on system size L in a power-law way, at variance with CFT.
Only those satisfying "neutrality" condition $\sum_i \boldsymbol{\lambda}_i = (-c_1, \ldots, -c_n) \equiv -\boldsymbol{\rho}_b$ have a chance to be described by CFT — but this is not guaranteed.

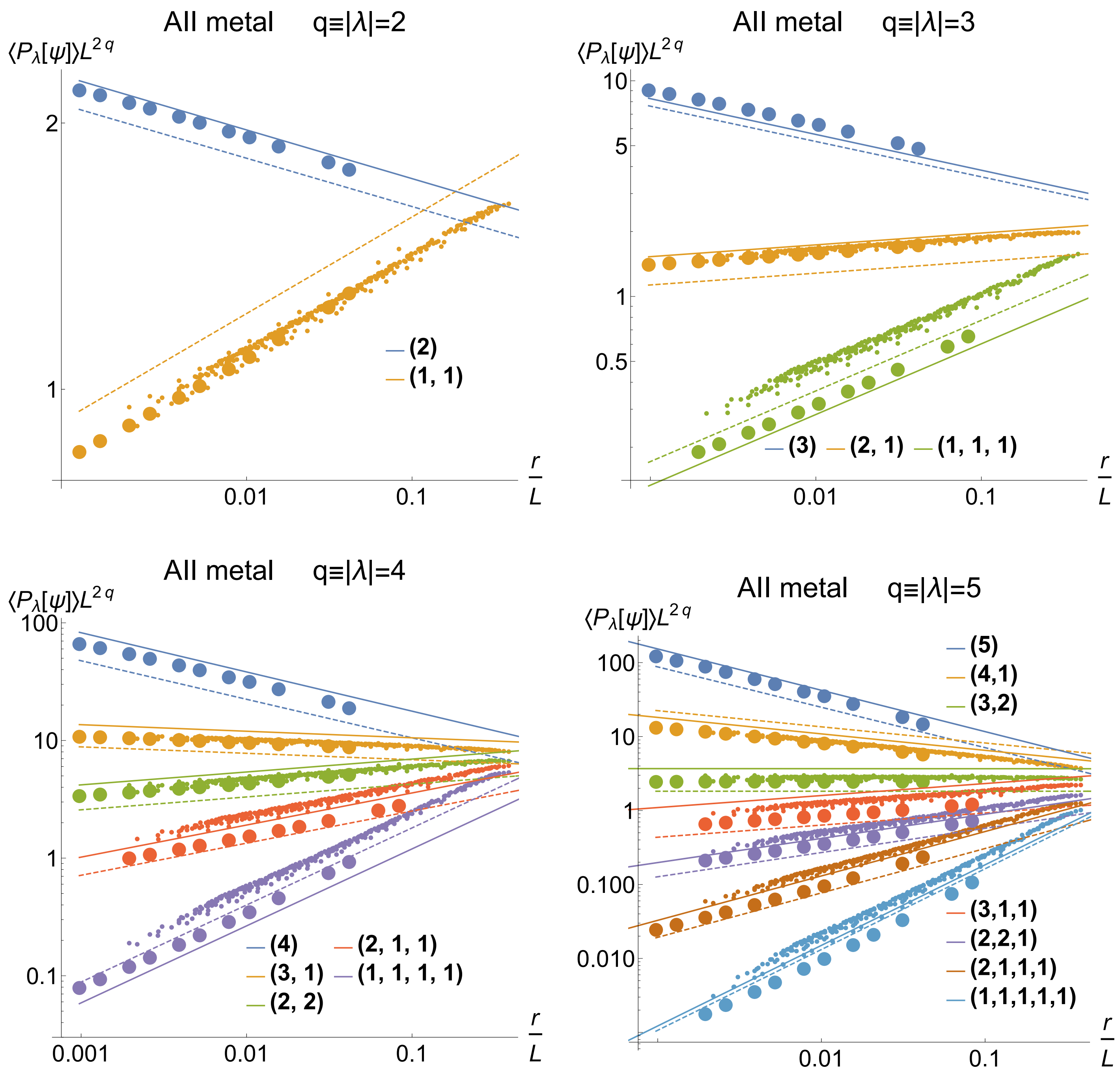
- We show that, if such correlation functions satisfy 2D conformal invariance,
x_λ is a quadratic function of q_j — "generalized parabolicity".
With Weyl symmetry $\longrightarrow$ single-parameter "generalized parabolicity"

$$x_\lambda^{\text{para}} \equiv x_{(q_1,q_2,\ldots)}^{\text{para}} = -b \sum_i q_i(q_i + c_i) \equiv -b\boldsymbol{\lambda} \cdot (\boldsymbol{\lambda} + \boldsymbol{\rho}_b) \equiv -b z_\lambda$$

z_λ– eigenvalues of the Laplace-Beltrami operator on the σ-model target space.
Generalized parabolicity — stringent test of conformal invariance!
In particular, violation of generalized parabolicity excludes
Wess-Zumino-Novikov-Witten models as candidates for critical theory.

More about this, including extension to higher d: talk by Ilya Gruzberg

Class AII. Metallic phase: Pure-scaling observables



dashed lines:

generalized parabolicity

$$x_\lambda^{\text{para}} = -b z_\lambda$$

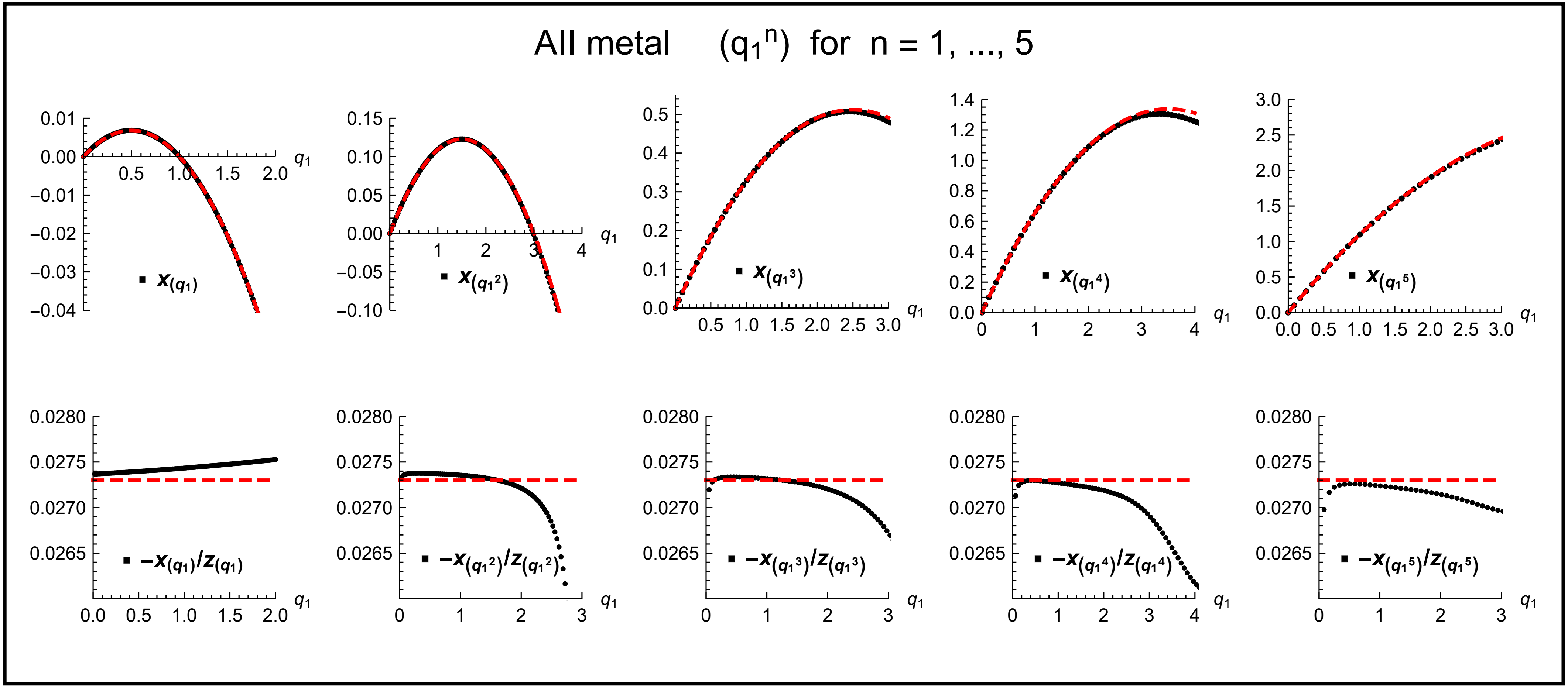
with $b = 0.0273$

Perfect confirmation of σ-model predictions:

- **pure-scaling observables**
- **generalized parabolicity**

(exact in one-loop order)

Class AII. Metallic phase



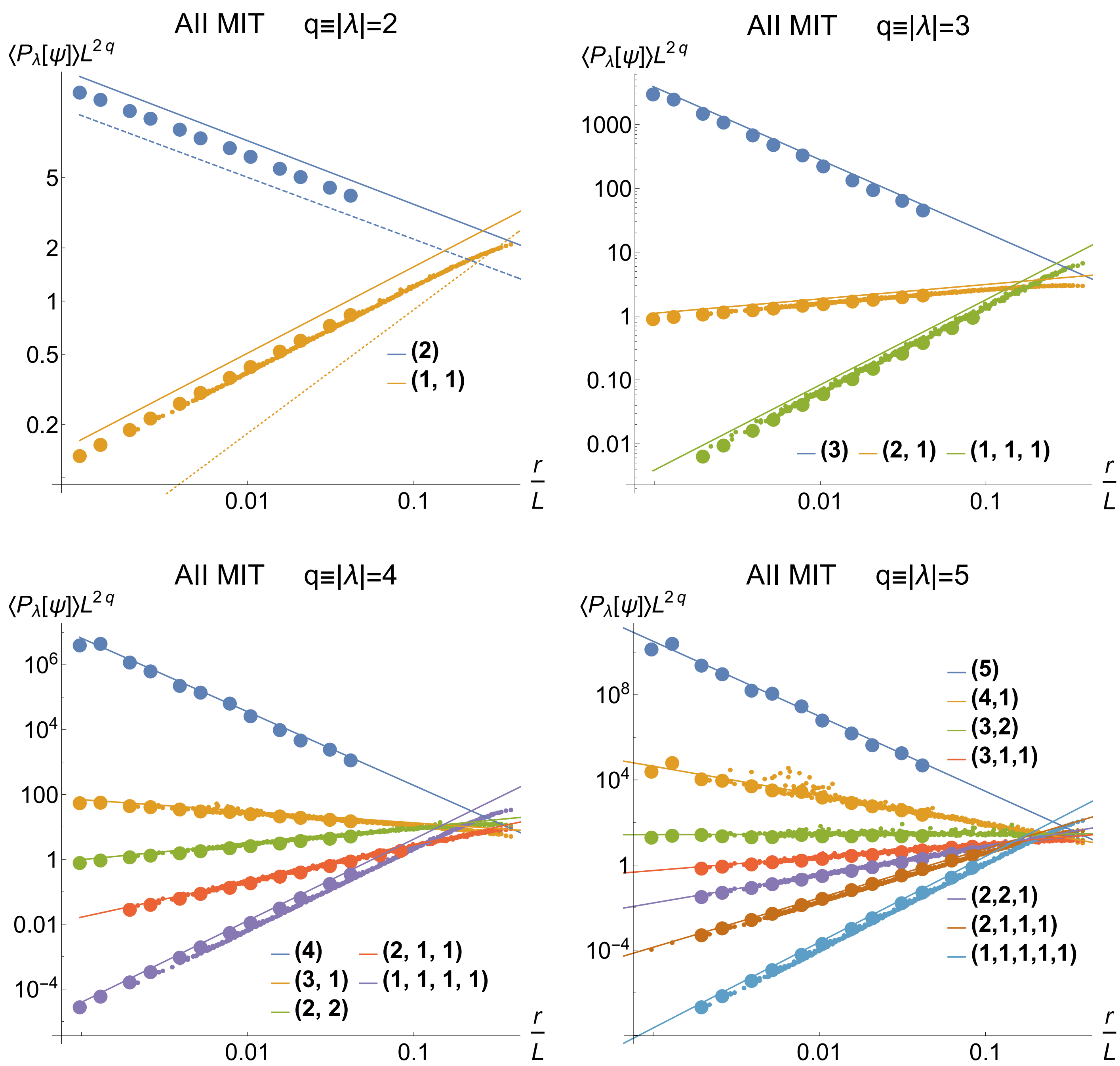
Red dashed lines: **generalized parabolicity $x_\lambda^{\rm para} = -bz_\lambda$ with $b = 0.0273$.**

Generalized parabolicity holds with an excellent accuracy, in consistency with analytical (σ-model) predictions. It is exact in one-loop order, and there is no two-loop and three-loop corrections in class AII.

Class AII. Scaling exponents x_λ

	rep. λ	x_λ^{MIT}	$x_\lambda^{\mathrm{MIT}}/b$	$x_\lambda^{\mathrm{metal}}$	$x_\lambda^{\mathrm{metal}}/b$	$x_\lambda^{\mathrm{para}}$
$q=2$	(2)	-0.361 ± 0.001	-2.08 ± 0.01	-0.0551 ± 0.0001	-2.017 ± 0.005	$-2b$
	(1,1)	0.489 ± 0.001	2.83 ± 0.01	0.1095 ± 0.0001	4.012 ± 0.005	$4b$
$q=3$	(3)	-1.14 ± 0.01	-6.57 ± 0.06	-0.1659 ± 0.0004	-6.08 ± 0.02	$-6b$
	(2,1)	0.225 ± 0.001	1.30 ± 0.01	0.0547 ± 0.0002	2.04 ± 0.01	$2b$
	(1,1,1)	1.333 ± 0.001	7.70 ± 0.01	0.3278 ± 0.0003	12.01 ± 0.01	$12b$
$q=4$	(4)	-2.27 ± 0.05	-13.13 ± 0.29	-0.334 ± 0.001	-12.21 ± 0.04	$-12b$
	(3,1)	-0.36 ± 0.01	-2.06 ± 0.06	-0.0557 ± 0.0005	-2.04 ± 0.02	$-2b$
	(2,2)	0.493 ± 0.005	2.85 ± 0.03	0.1095 ± 0.0005	4.01 ± 0.02	$4b$
	(2,1,1)	1.111 ± 0.003	6.42 ± 0.02	0.2728 ± 0.0005	9.99 ± 0.02	$10b$
	(1,1,1,1)	2.515 ± 0.002	14.54 ± 0.01	0.6545 ± 0.0003	23.97 ± 0.01	$24b$
$q=5$	(5)	-3.52 ± 0.09	-20.37 ± 0.17	-0.559 ± 0.003	-20.48 ± 0.52	$-20b$
	(4,1)	-1.35 ± 0.07	-7.82 ± 0.40	-0.223 ± 0.001	-8.16 ± 0.04	$-8b$
	(3,2)	0.02 ± 0.02	0.08 ± 0.12	-0.0006 ± 0.0009	0.02 ± 0.03	0
	(3,1,1)	0.64 ± 0.01	3.67 ± 0.06	0.1623 ± 0.0008	5.95 ± 0.03	$6b$
	(2,2,1)	1.333 ± 0.005	7.70 ± 0.03	0.327 ± 0.0008	11.97 ± 0.03	$12b$
	(2,1,1,1)	2.316 ± 0.004	13.39 ± 0.02	0.5997 ± 0.0005	21.99 ± 0.02	$22b$
	(1,1,1,1,1)	4.031 ± 0.004	23.30 ± 0.02	1.0895 ± 0.0004	39.91 ± 0.02	$40b$

Class AII. Anderson transition: Pure-scaling observables



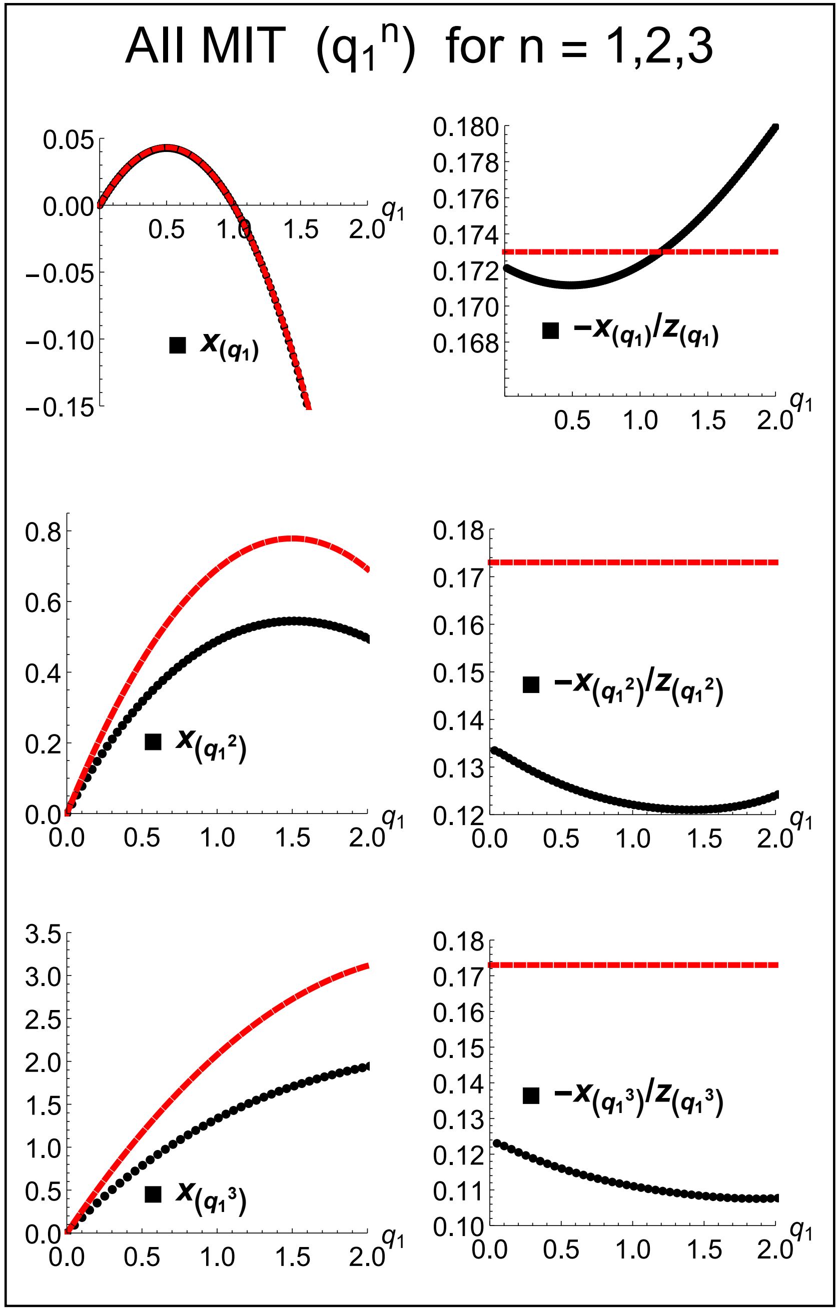
Perfect confirmation of σ-model predictions:

- pure-scaling observables
- Weyl symmetries

(see next two slides)

generalized parabolicity strongly violated (see also next slide)

Class AII. Anderson-transition critical point



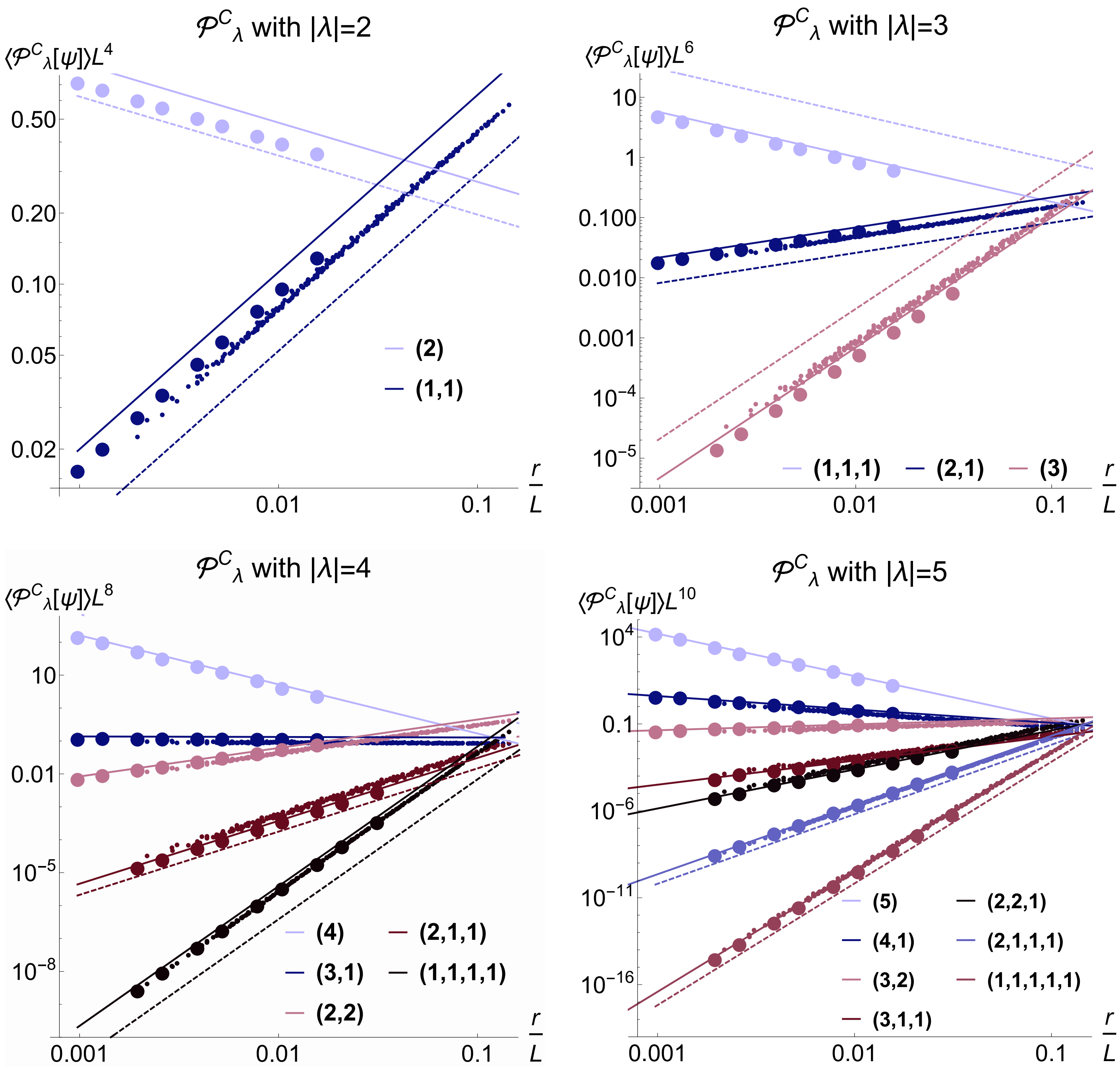
- Weyl symmetry holds nicely (see also the table on the next slide)
- Generalized parabolicity (red lines) strongly violated

$\longrightarrow$ violation of conformal invariance

Class AII. Scaling exponents x_λ

	rep. λ	x_λ^{MIT}	x_λ^{MIT}/b	x_λ^{metal}	$x_\lambda^{\text{metal}}/b$	x_λ^{para}
$q = 2$	(2)	-0.361 ± 0.001	-2.08 ± 0.01	-0.0551 ± 0.0001	-2.017 ± 0.005	$-2b$
	(1,1)	0.489 ± 0.001	2.83 ± 0.01	0.1095 ± 0.0001	4.012 ± 0.005	$4b$
$q = 3$	(3)	-1.14 ± 0.01	-6.57 ± 0.06	-0.1659 ± 0.0004	-6.08 ± 0.02	$-6b$
	(2,1)	0.225 ± 0.001	1.30 ± 0.01	0.0547 ± 0.0002	2.04 ± 0.01	$2b$
	(1,1,1)	1.333 ± 0.001	7.70 ± 0.01	0.3278 ± 0.0003	12.01 ± 0.01	$12b$
$q = 4$	(4)	-2.27 ± 0.05	-13.13 ± 0.29	-0.334 ± 0.001	-12.21 ± 0.04	$-12b$
	(3,1)	-0.36 ± 0.01	-2.06 ± 0.06	-0.0557 ± 0.0005	-2.04 ± 0.02	$-2b$
	(2,2)	0.493 ± 0.005	2.85 ± 0.03	0.1095 ± 0.0005	4.01 ± 0.02	$4b$
	(2,1,1)	1.111 ± 0.003	6.42 ± 0.02	0.2728 ± 0.0005	9.99 ± 0.02	$10b$
	(1,1,1,1)	2.515 ± 0.002	14.54 ± 0.01	0.6545 ± 0.0003	23.97 ± 0.01	$24b$
$q = 5$	(5)	-3.52 ± 0.09	-20.37 ± 0.17	-0.559 ± 0.003	-20.48 ± 0.52	$-20b$
	(4,1)	-1.35 ± 0.07	-7.82 ± 0.40	-0.223 ± 0.001	-8.16 ± 0.04	$-8b$
	(3,2)	0.02 ± 0.02	0.08 ± 0.12	-0.0006 ± 0.0009	0.02 ± 0.03	0
	(3,1,1)	0.64 ± 0.01	3.67 ± 0.06	0.1623 ± 0.0008	5.95 ± 0.03	$6b$
	(2,2,1)	1.333 ± 0.005	7.70 ± 0.03	0.327 ± 0.0008	11.97 ± 0.03	$12b$
	(2,1,1,1)	2.316 ± 0.004	13.39 ± 0.02	0.5997 ± 0.0005	21.99 ± 0.02	$22b$
	(1,1,1,1,1)	4.031 ± 0.004	23.30 ± 0.02	1.0895 ± 0.0004	39.91 ± 0.02	$40b$

Class C. SQH transition: Pure-scaling observables



excellent agreement with analytical results from percolation mapping (dashed lines)

Perfect confirmation of σ-model predictions:

- **pure-scaling observables**
- **Weyl symmetries**

generalized parabolicity strongly violated

(see also next three slides)

SQH transition and classical percolation

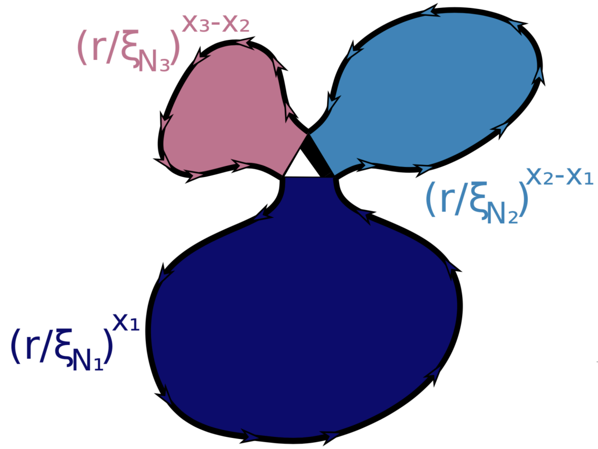
Classical percolation: Probability that n hull segments come close at two points separated by a distance r:

$$P_n(r) \sim r^{-x_n^{\rm h}}(p-p_c)^{\nu x_n^{\rm h}}, \qquad r \lesssim \xi = (p-p_c)^{-\nu}$$ Saleur, Duplantier, 1987

n-hull exponents $\quad x_n^{\rm h} = \dfrac{4n^2-1}{12}, \qquad n = 1, 2, \ldots$

In particular, for n = 1, 2, and 3: $\quad x_1^{\rm h} = \dfrac{1}{4}, \quad x_2^{\rm h} = \dfrac{5}{4}, \quad x_3^{\rm h} = \dfrac{35}{12}$

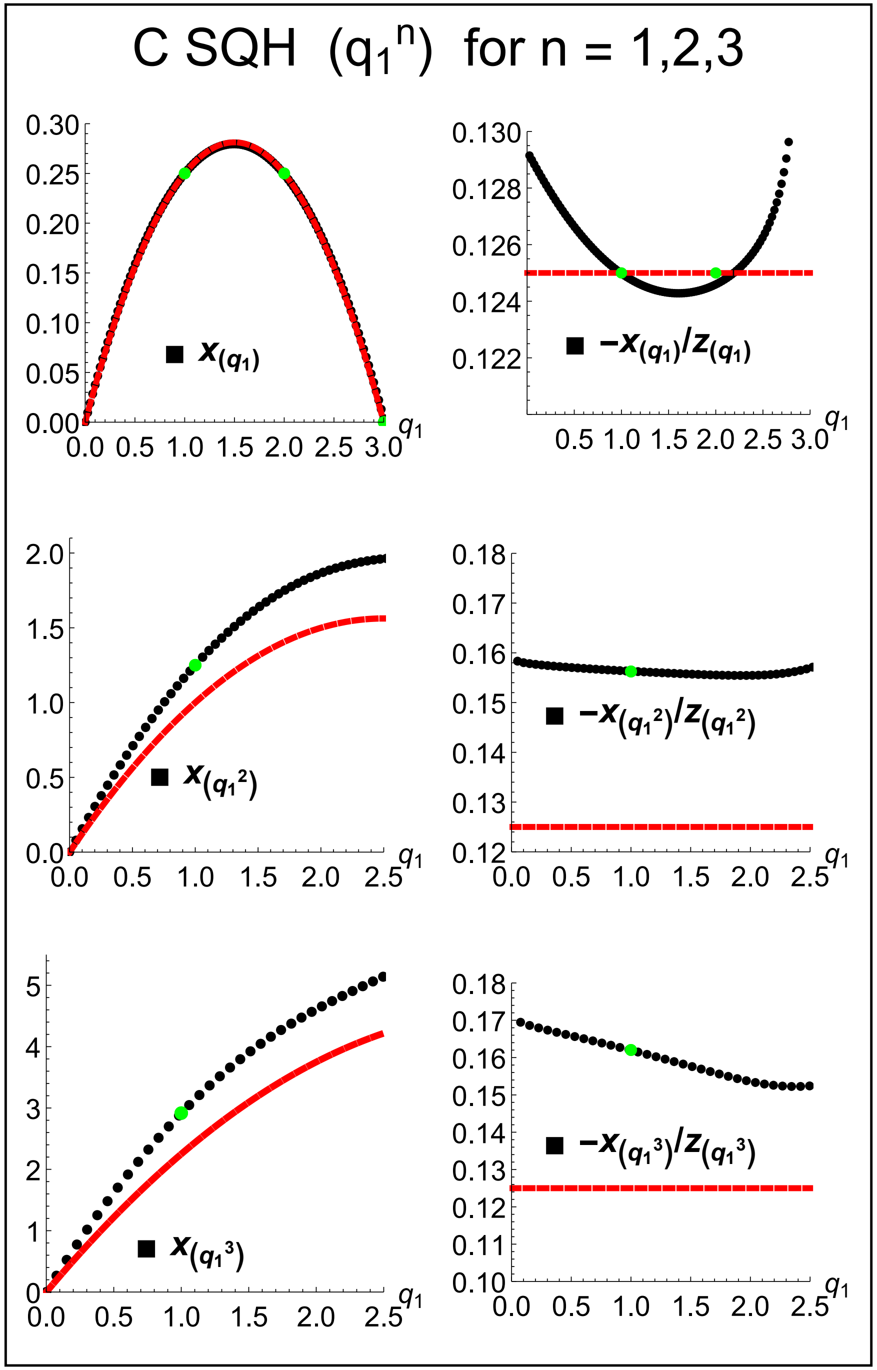
Mapping of SQH transition to percolation for a certain class of correlation functions



Exact results for a subset of SQH exponents x_λ: $\quad \boxed{x_{(1^n)} = x_n^{\rm h}}$

$$x_{(1)} = x_{(2)} = \frac{1}{4}, \qquad x_{(1,1)} = x_{(2,1)} = \frac{5}{4}, \qquad x_{(1,1,1)} = x_{(2,1,1)} = \frac{35}{12}, \quad \text{etc}$$

Class C.   SQH transition.



- Excellent agreement of numerical values with analytical results (from mapping to percolation; green symbols)
- Weyl symmetry holds nicely
- Generalized parabolicity (red lines) strongly violated

$\longrightarrow$ violation of conformal invariance

SQH transition (class C). Scaling exponents x_λ

	λ	x_λ^{perc}	x_λ^{num}	x_λ^{para}
$q=1$	(1)	$x_1^{\text{h}} = 1/4 = 0.25$	—	$1/4$
$q=2$	(2)	$x_1^{\text{h}} = 1/4 = 0.25$	0.249 ± 0.001	$1/4$
	(1,1)	$x_2^{\text{h}} = 5/4 = 1.25$	1.251 ± 0.001	1
$q=3$	(3)	0	0.004 ± 0.004	0
	(2,1)	$x_2^{\text{h}} = 5/4 = 1.25$	1.249 ± 0.002	1
	(1^3)	$x_3^{\text{h}} = 35/12 \simeq 2.917$	2.915 ± 0.002	$9/4$
$q=4$	(4)	—	-0.49 ± 0.02	$-1/2$
	(3,1)	—	0.985 ± 0.007	$3/4$
	(2,2)	—	1.865 ± 0.006	$3/2$
	(2,1,1)	$x_3^{\text{h}} = 35/12 \simeq 2.917$	2.911 ± 0.005	$9/4$
	(1^4)	$x_4^{\text{h}} = 21/4 = 5.25$	5.242 ± 0.004	4
$q=5$	(5)	—	-1.19 ± 0.06	$-5/4$
	(4,1)	—	0.48 ± 0.03	$1/4$
	(3,2)	—	1.59 ± 0.02	$5/4$
	(3,1,1)	—	2.64 ± 0.02	2
	(2,2,1)	—	3.50 ± 0.02	$11/4$
	$(2,1^3)$	$x_4^{\text{h}} = 21/4 = 5.25$	5.23 ± 0.01	4
	(1^5)	$x_5^{\text{h}} = 33/4 = 8.25$	8.16 ± 0.01	$25/4$

- Excellent agreement of numerical values x_λ^{num} with analytical results x_λ^{perc} (from mapping to percolation)
- Weyl symmetry holds nicely
- Generalized parabolicity (x_λ^{para}, last column) strongly violated

$\longrightarrow$ violation of conformal invariance

Extensions to other 2D critical points / symmetry classes

- **quantum Hall transition (class A)**

- **metallic phases and MIT in classes D and DIII**

  Peculiarity: two disjoint components of the σ-model manifold

  $\longrightarrow$ domain walls $\longrightarrow$ violation of Weyl symmetry at the MIT

- **chiral classes AIII, BDI, CII (models with sublattice symmetry): critical-metal phases and metal-insulator transitions**

  Peculiarities:

  (i) Observables / exponents labeled by a pair of multi-indices:

  $\lambda = (q_1, \ldots, q_n)$ and $\bar{\lambda} = (\bar{q}_1, \ldots, \bar{q}_{\bar{n}})$ corresponding to two sublattices

  (ii) peculiar Weyl group (only permutations) $\longrightarrow$ affects Weyl symmetries

  $\lambda = \lambda'$ observables: $x_{\lambda,\lambda} = x_{w(\lambda),w(\lambda)}$ $\quad w \in$ conventional Weyl group

- **(work in progress)**

  topologically protected critical points at surfaces of topological superconductors or in models of Dirac fermions (classes CI, DIII, AIII)

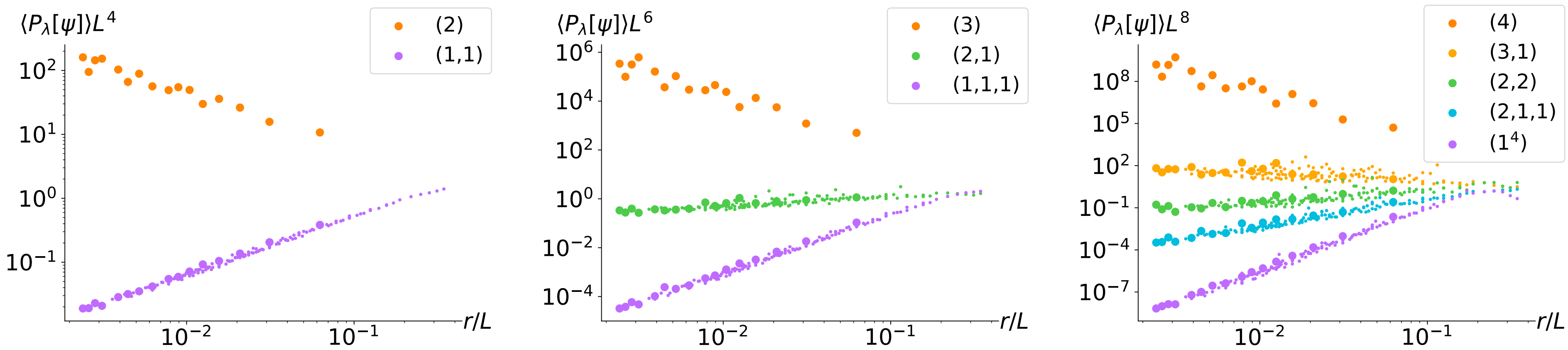
  $\longrightarrow$ WZNW theories $\longrightarrow$ generalized parabolicity,

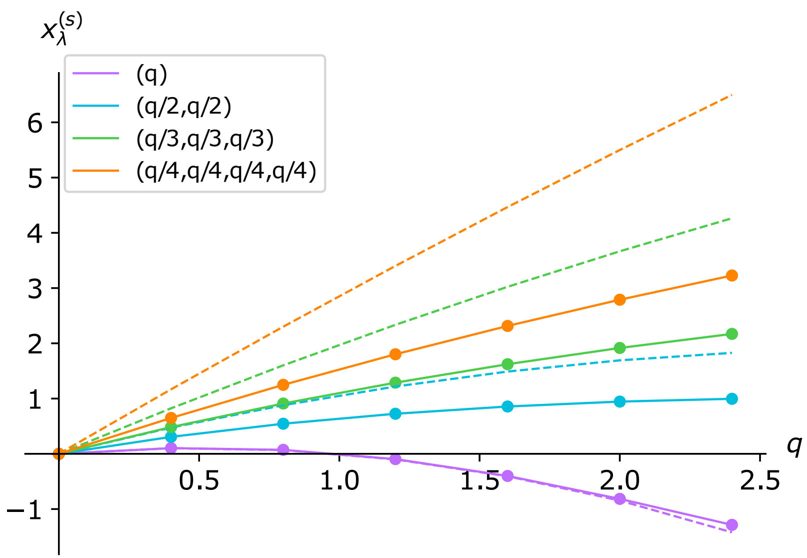
  remains to be verified numerically

Surface generalized multifractality

Analytics: Sigma-model analysis extended to observables near the boundary. Construction of observable and Weyl symmetries keep their form.

Class-AII Anderson transition.





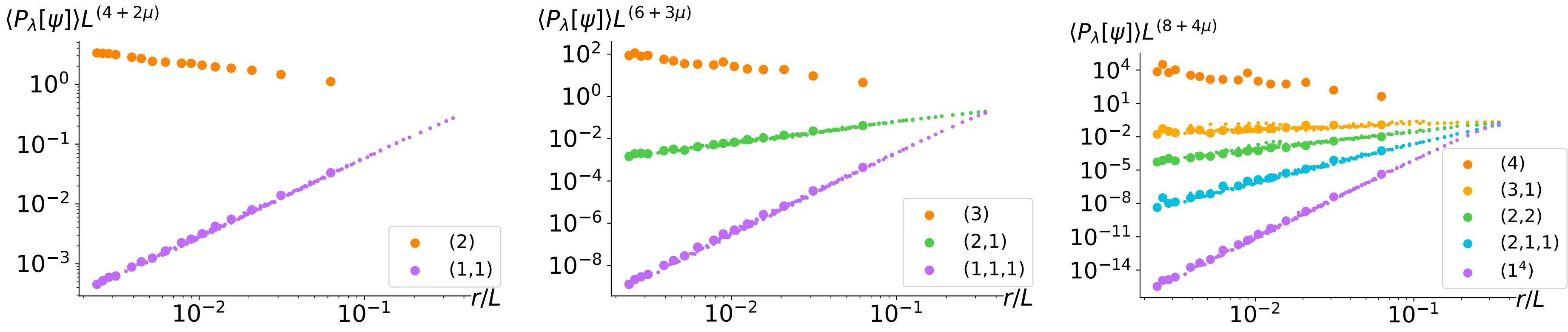
Observables – OK, Weyl symmetries – OK

$\longrightarrow$ confirmation of validity of sigma-model approach also at boundary

strong violation of generalized parabolicity, corroborates corresponding bulk fundings

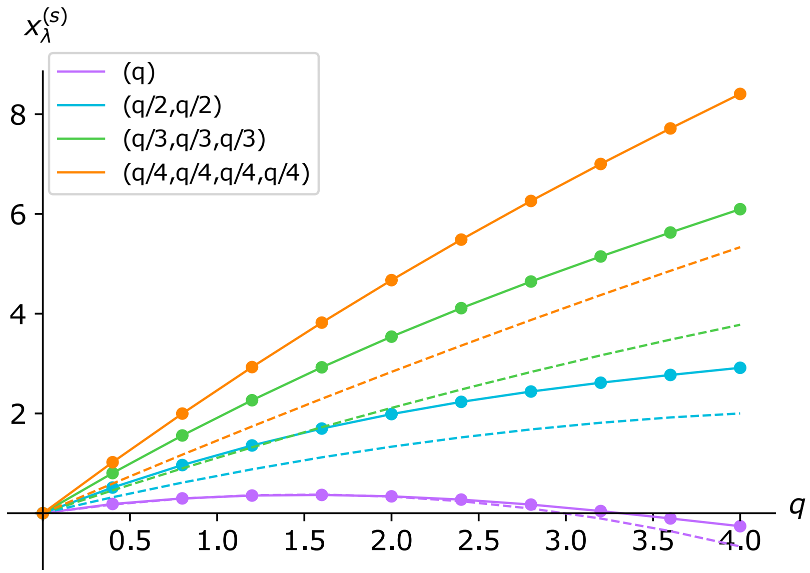
$\longrightarrow$ violation of conformal invariance

Surface generalized multifractality: SQH transition (class C)



Observables – OK, Weyl symmetries – OK

$\longrightarrow$ **confirmation of validity of sigma-model approach also at boundary**



strong violation of generalized parabolicity,
corroborates corresponding bulk fundings

$\longrightarrow$ **violation of conformal invariance**

IQH transition (class A): similar results

Surface generalized multifractality: SQH transition (class C)
Analytical results from percolation mapping

Mapping $\longrightarrow$ some exponents $x_\lambda^{(s)}$ can be expressed in terms of percolation n-hull boundary exponents, which were calculated by Saleur, Bauer, 1989

$$\longrightarrow \quad x_{(1^n)}^{(s)} = \frac{n(2n-1)}{3}, \quad n = 1, 2, 3, \ldots \quad \text{Weyl symmetry} \longrightarrow \quad x_{(2,1^{n-1})}^{(s)} = x_{(1^n)}^{(s)}$$

λ	$\tau_\lambda^{(s)}$	$\tau_{\lambda,\mathrm{perc}}^{(s)}$	$\Delta_\lambda^{(s)}$	$x_\lambda^{(s)}$	$x_{\lambda,\mathrm{perc}}^{(s)}$
(1)	1.0815	13/12	-0.0018	0.3315 ± 0.0022	1/3
(2)	2.838	17/6	-0.329	0.338 ± 0.009	1/3
$(1,1)$	4.487	4.5	1.320	1.987 ± 0.007	2
(3)	4.36	4.25	-0.89	0.11 ± 0.05	0
$(2,1)$	6.27	6.25	1.02	2.02 ± 0.03	2
$(1,1,1)$	9.15	9.25	3.90	4.90 ± 0.04	5
(4)	5.74	-	-1.59	-0.26 ± 0.18	-
$(3,1)$	7.86	-	0.52	1.86 ± 0.09	-
$(2,2)$	8.92	-	1.58	2.92 ± 0.05	-
$(2,1,1)$	10.83	11	3.49	4.83 ± 0.12	5
$(1,1,1,1)$	14.41	46/3	7.07	8.41 ± 0.06	28/3

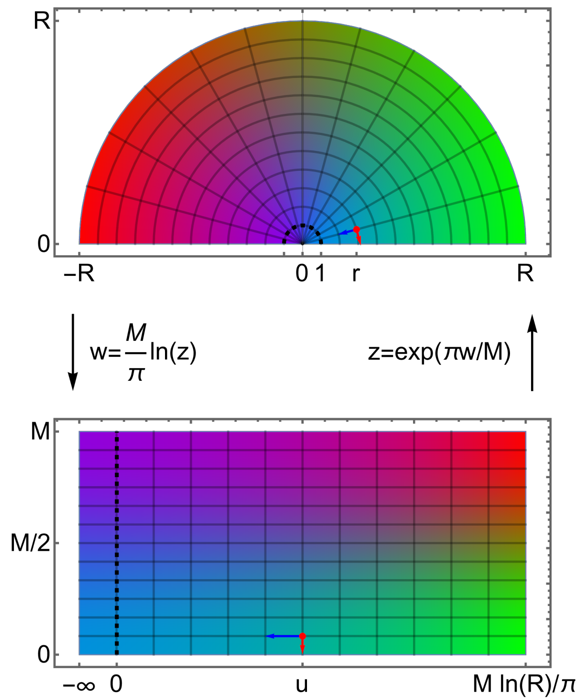
Excellent agreement between numerical and analytical values of exponents!

Logarithmic conformal mapping: 2D to quasi-1D

Logarithmic (exponential) mapping 2D (z) $\longleftrightarrow$ quasi-1D (w)

semicircle of radius R $\longleftrightarrow$ strip of width M and length $L = \frac{M}{\pi} \ln R$

$$w = \frac{M}{\pi} \ln z, \qquad z = \exp\left(\frac{\pi}{M} w\right)$$



Assume invariance with respect to this mapping

$$\longrightarrow \quad \pi \frac{dx^{(s)}_{(q^n)}}{dq}\bigg|_{q=0} = 2M \sum_{i=1}^{n} \mathcal{L}_i \qquad n = 1,2,3,\ldots$$

$\mathcal{L}_i$ – Lyapunov exponents

Numerical data are in very good agreement with these relations

	$\pi \frac{dx^{(s)}_{(q^n)}}{dq}\Big\vert_{q=0}$	$2M\sum_{i=1}^{n}\mathcal{L}_i$
$n=1$	1.337 ± 0.020	1.331 ± 0.005
$n=2$	5.42 ± 0.03	5.39 ± 0.02
$n=3$	12.12 ± 0.05	12.05 ± 0.06
$n=4$	21.18 ± 0.07	21.25 ± 0.14

(data for class-AII MIT presented; similar results for SQH and IQH transitions)

invariance with respect to exponential map although full conformal invariance is violated

Summary

- **Generalized multifractality of wave functions at Anderson transitions**
- **Pure-scaling composite operators in the σ-model formalism**
- **Construction of pure-scaling eigenfunction observables for all symmetry classes and its numerical verification**
- **Symmetries of scaling exponents: Weyl-group invariance**
- **Analytical evaluation of a certain subset of generalized-multifractality exponents for SQH transition via mapping to percolation**
- **Numerical evaluation of generalized-multifractality exponents for 2D critical points of various symmetry classes**
- **Excellent agreement with predictions of σ models on pure-scaling observables, Weyl symmetries, generalized parabolicity in the metallic phase**
  $\longrightarrow$ confirmation of σ models as field theories of Anderson localization
- **Violation of generalized parabolicity—and thus of conformal invariance— at SQH transition and several other 2D Anderson-localization critical points. Excludes Wess-Zumino-Novikov-Witten models as critical theories.**
- **Surface generalized multiftactality.**
- **Invariance with respect to 2D $\longleftrightarrow$ quasi-1D logarithmic mapping**

Outlook : **models with interaction; experiment; . . .**