Inverted pendulum driven by a random force: statistics of non-falling trajectories & supersymmetry

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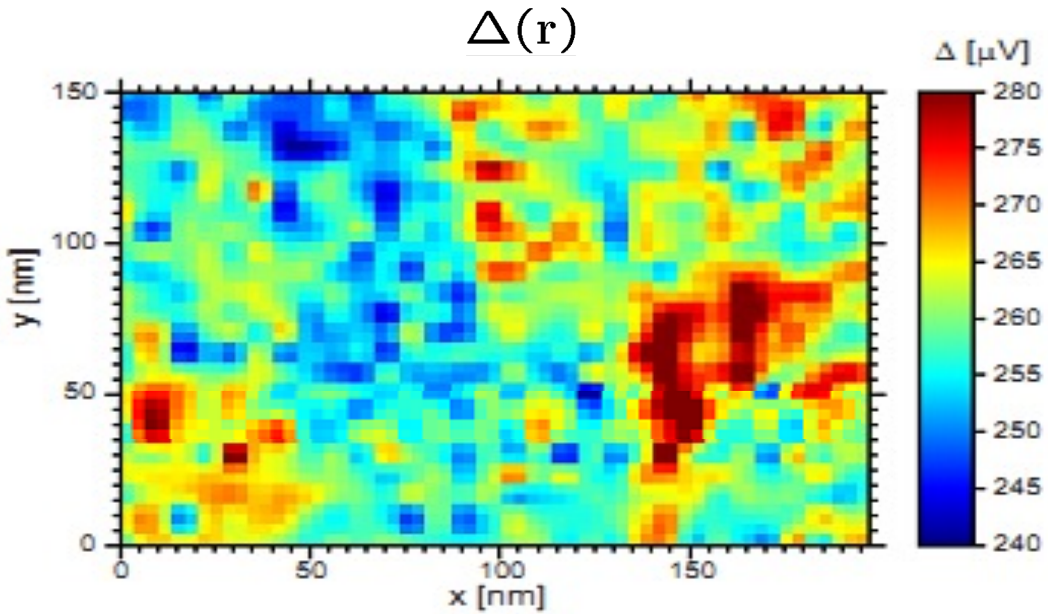
Inhomogeneous 1D superconducting wires



- **Experimental fact**

  STM measurements
  on TiN films
  [B. Sacepe *et al* (2008)]

  $k_F l \gtrsim 1$

- **Theoretical description** based on the Usadel equation

$$(D/2)d^2\theta/dx^2 + iE\sin\theta + \Delta(\mathbf{r})\cos\theta = 0$$

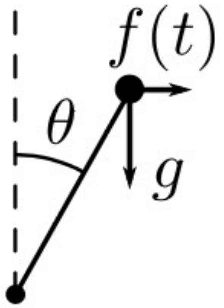
$$\nu(E,\mathbf{r}) = \nu_0 \,\mathrm{Re}\cos\theta(E,\mathbf{r}) \qquad -\pi/2 < \mathrm{Re}\,\theta < \pi/2$$

- **In 1D wires**, Usadel equation describes dynamics of a driven pendulum with complex frequency $\longrightarrow$ instability $\longrightarrow$ problems with negative DOS

Outline

- Whitney's problem
- Stochastic Whitney's problem
  - Weak driving
  - Field-theoretcal appoach
  - Statistics of the never-falling trajectory
- Lyapunov exponent
- Physical interpretation

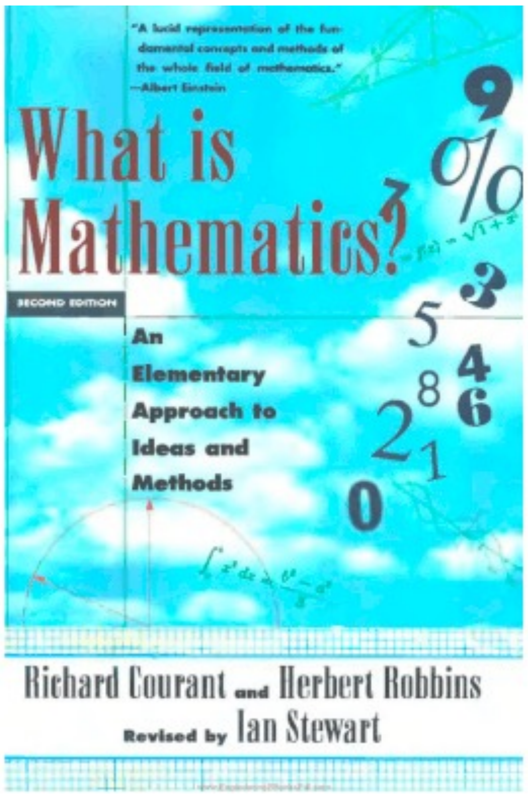
Inverted pendulum under horizontal driving



$$\ddot{\theta} = \omega^2 \sin\theta + f(t)\cos\theta$$

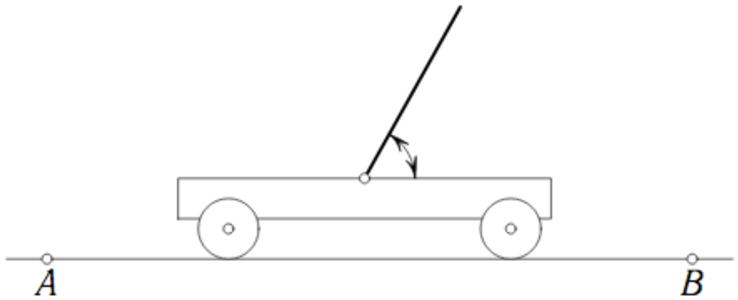
$$-\pi/2 < \theta < \pi/2$$

H. Whitney's problem



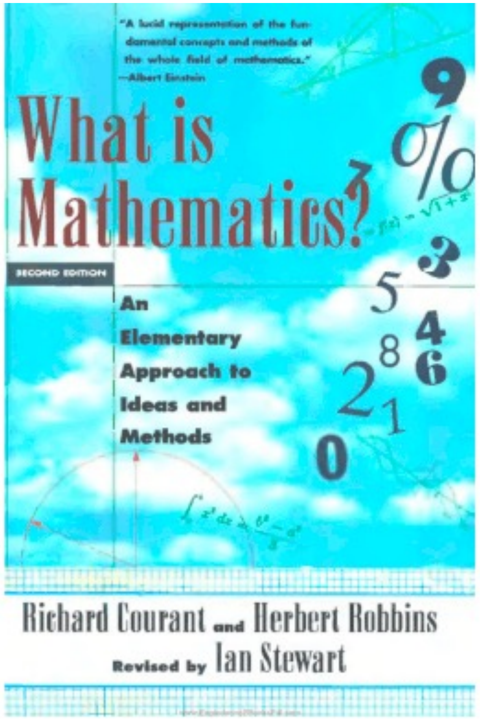
Courant & Robbins (1941)

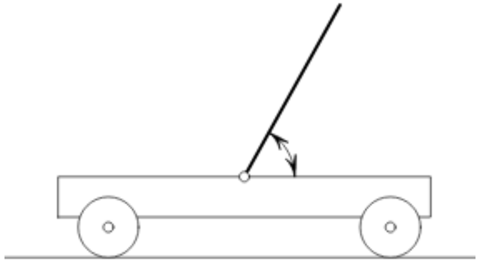
Is it possible to place the rod in such a position that, if it is released at the instant when the train starts and allowed to move solely under the influence of gravity and the motion of the train, it will not fall to the floor during the entire journey from A to B?

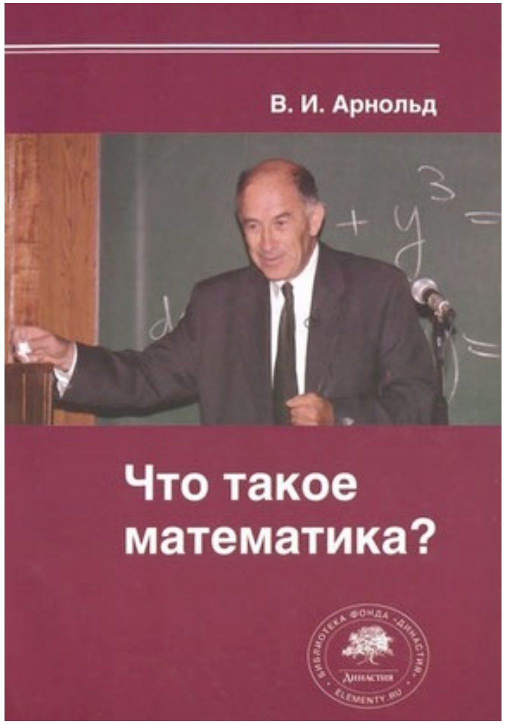


It might seem quite unlikely that for any given schedule of motion the interplay of gravity and reaction forces will always permit such a maintenance of balance under the single condition that the initial position of the rod is suitably chosen. Yet we state that such a position always exists.

Whitney's problem & Arnold







Courant & Robbins (1941)

Proof based on continuity

Arnold (2002)

Continuity to be proved

Paradoxical as this assertion might seem at first sight, it can be proved easily once one concentrates on its essentially topological character. No detailed knowledge of the laws of dynamics is needed; only the following simple assumption of a physical nature need be granted: *The motion of the rod depends continuously on its initial position.*

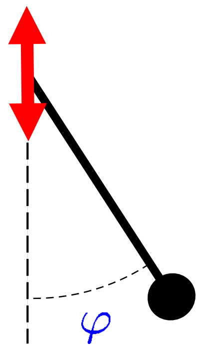
Whitney's problem in historical perspective

- R. Courant and H. Robbins (1941)
  Continuity arguments
- A. Broman (1958)
  Sets of initial conditions leading to first crossing $\pi/2$ or $-\pi/2$ are open
- T. Poston (1976)
  Claims a non-falling trajectory may not exist
- V. Arnold (2002)
  Existence still to be proven
- I. Polekhin (2014)
  Proof using Wazewski topological principle
- S. Bolotin and V. Kozlov (2015)
  Other topological arguments
- O. Zubelevich (2015)
  Alternative approach + review
- A. Shen (2019) *Rod in a train: a mechanical problem of H. Whitney, or Much Ado About Nothing*, arXiv:1907.01598

Kapitza pendulum

- Vertical driving

$$\ddot{\varphi} = -\omega^2 \sin\varphi + f(t)\sin\varphi$$



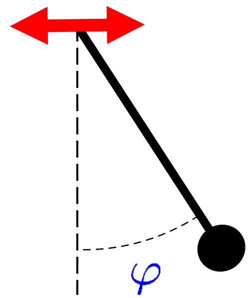
periodic driving

$$f(t) \propto a\cos\gamma t$$

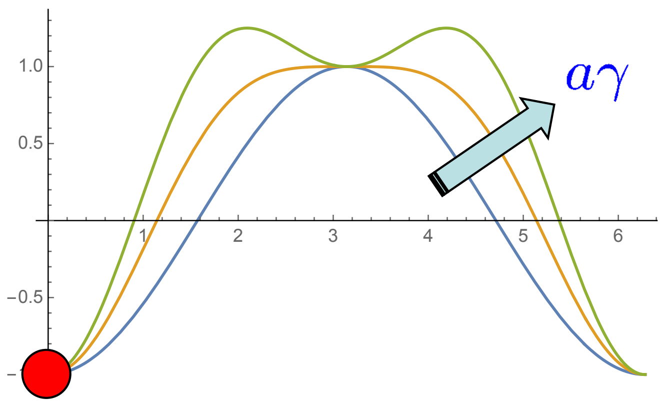
$$\gamma \gg \omega$$

- Horizontal driving

$$\ddot{\varphi} = -\omega^2 \sin\varphi + f(t)\cos\varphi$$



$$U_{эф} = mgl\left(-\cos\varphi + \frac{a^2\gamma^2}{4gl}\sin^2\varphi\right)$$



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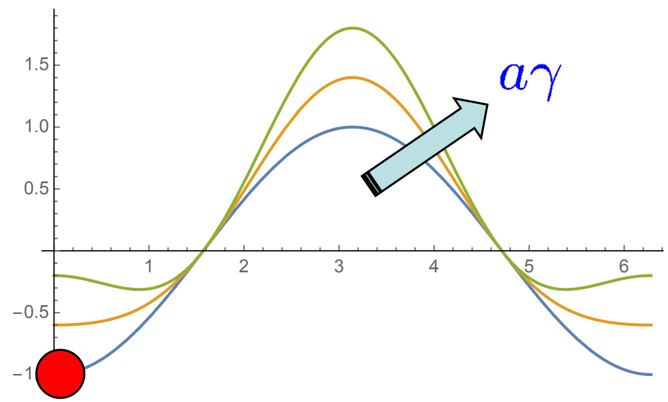
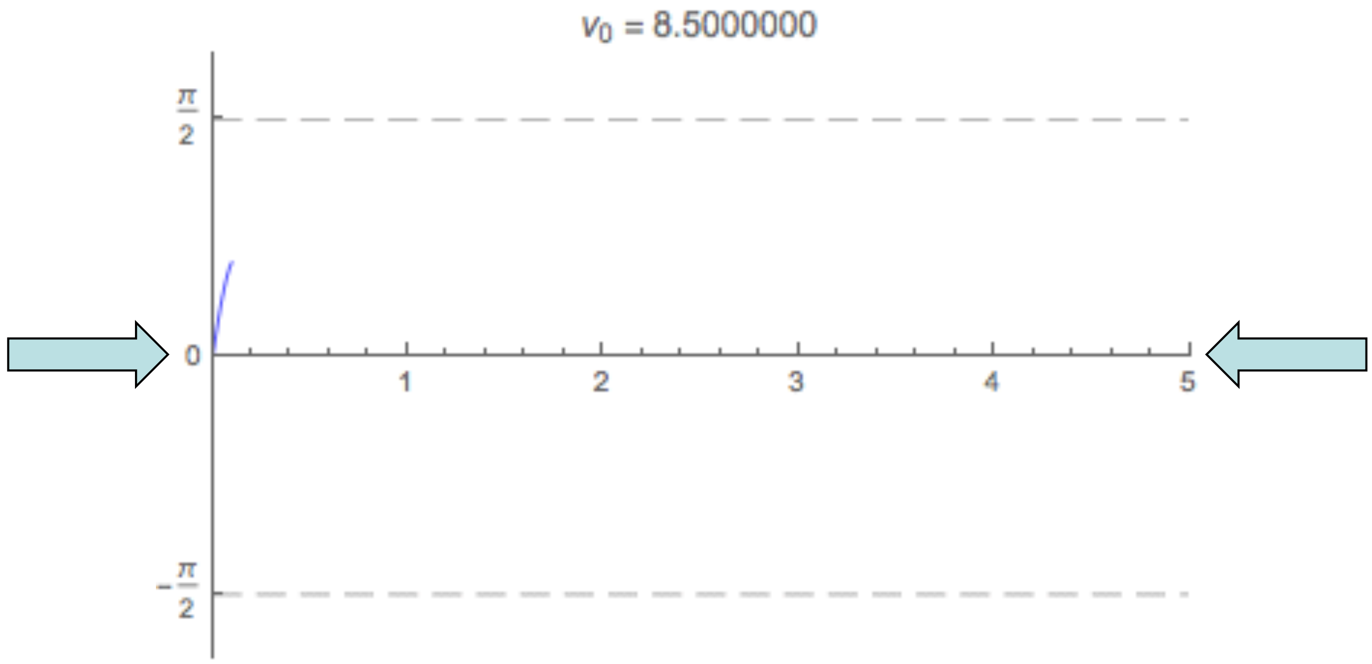


Warm up: numerics

$$\ddot{\theta} = \omega^2 \sin\theta + f(t)\cos\theta$$

$$-\pi/2 < \theta < \pi/2$$



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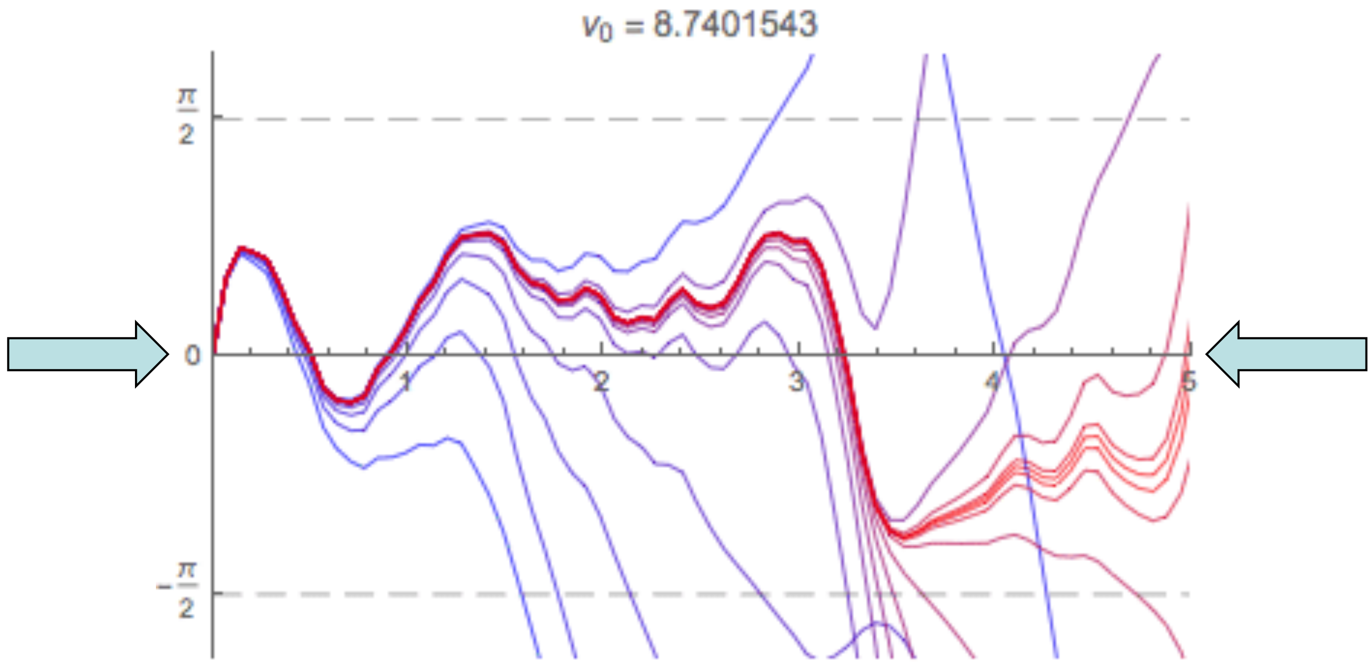
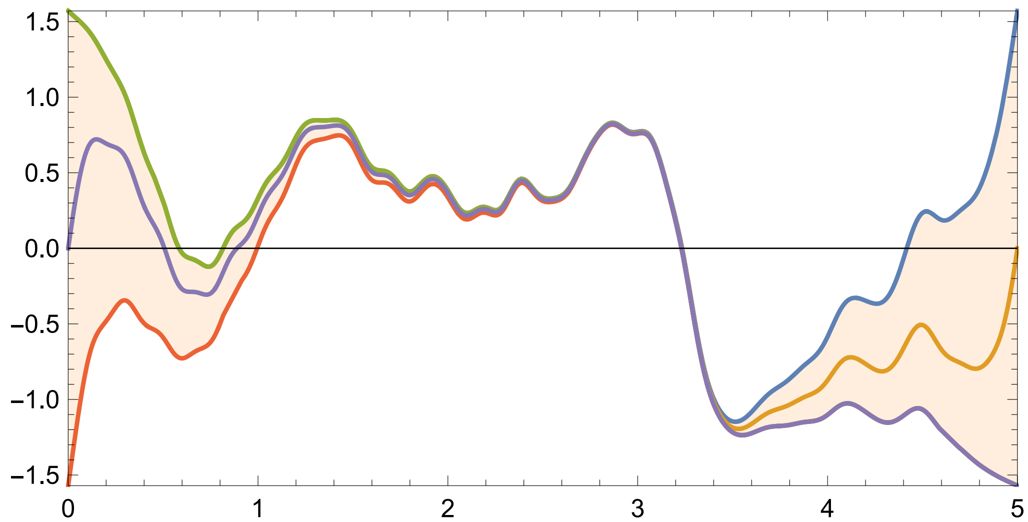


Non-falling trajectory

$$\ddot{\theta} = \omega^2 \sin\theta + f(t)\cos\theta$$

$$-\pi/2 < \theta < \pi/2$$



- $\exists$ a **unique** non-falling trajectory with $\theta(0) = \theta_1$ to $\theta(T) = \theta_2$ for all T.
- At $T \to \infty$ this non-falling trajectory turns to a unique never-falling trajectory
- Exponentially unstable attractor

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Problem formulation

$$\ddot{\theta} = \omega^2 \sin\theta + f(t)\cos\theta$$

$$-\pi/2 < \theta < \pi/2$$

$$\langle f(t) f(t') \rangle = 2\,\alpha\,\delta(t - t')$$

Describe statistics of the non-falling trajectory and calculate the distribution function $P(\theta, \dot{\theta})$

The only dimensionless parameter:

$$\alpha/\omega^3$$

Weak-driving limit

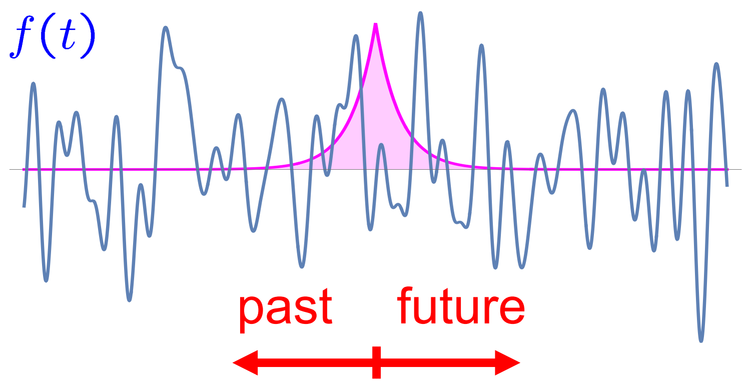
- If $\alpha/\omega^3 \ll 1$ then the pendulum equation can be linearized ($\theta \ll 1$):

$$\ddot{\theta} = \omega^2\theta + f(t)$$

- Linear equation with additive noise
- Bounded solution is unique and can be written explicitly:

$$\theta(t) = \int_{-\infty}^{\infty} G(t-t')f(t')\,dt'$$

$$G(t) = -\frac{1}{2\omega}\exp(-\omega|t|)$$



- Probability distribution function:

$$P(\theta,\dot{\theta}) = \frac{\omega^2}{\pi\alpha}\exp\left(-\frac{\omega^3}{\alpha}\theta^2 - \frac{\omega}{\alpha}\dot{\theta}^2\right)$$

$$P(\theta) = \sqrt{\frac{\omega^3}{\pi\alpha}}\exp\left(-\frac{\omega^3}{\alpha}\theta^2\right)$$

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Field theory

- Classical nonlinear dynamic equation with randomness

  Martin, Siggia & Rose (1973)
  Halperin, Hohenberg & Ma (1974)
  Parisi & Sourlas (1979)

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Random Magnetic Fields, Supersymmetry, and Negative Dimensions

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Istituto Nazionale di Fisica Nucleare, Frascati, Italy

and

N. Sourlas
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure, 75231 Paris Cédex 05, France
(Received 26 June 1979)

Parisi & Sourlas approach

$$\ddot{\theta} = F(\theta, t) \qquad F(\theta, t) = \omega^2 \sin\theta + f(t)\cos\theta$$

- Partition function (number of solutions)

$$Z = \int \mathcal{D}\theta(t)\,\delta(\theta(t) - \theta_{\text{sol}}(t))$$

$$Z = \int \mathcal{D}\theta(t)\,\delta(-\ddot{\theta} + F(\theta))\,\left|\det\left(-\partial_t^2 + F'(\theta)\right)\right|$$

- Removing the absolute value due to uniquness of the NFT **!**

$$Z = \int D\theta\, D\lambda\, D\overline{\chi}\, D\chi \exp\left[\int dt \left\{ i\lambda(-\ddot{\theta} + F(\theta)) + \overline{\chi}(-\partial_t^2 + F'(\theta))\chi \right\}\right]$$

kinetic energy: $i\dot{\lambda}\dot{\theta} + \dot{\overline{\chi}}\dot{\chi}$

- Disorder averaging generates a local action and couples bosons to fermions

$\langle f(t) f(t') \rangle = 2\alpha\delta(t - t')$

Transfer-matrix Hamiltonian

Efetov & Larkin (1983)

- From a 1D field theory to an effective quantum mechanics:

$$\partial_t \hat{\Psi} = -\mathcal{H}\hat{\Psi} \qquad \hat{\Psi} = \hat{\Psi}(\theta, \lambda, \overline{\chi}, \chi) = \Psi(\theta, \lambda) + \Phi(\theta, \lambda)\overline{\chi}\chi$$

- Fourier transform w.r.t. the decoupler field λ (p corresponds to $\partial_t \theta$):

$$\hat{\Psi}(\theta, \lambda) = \int \hat{\Psi}(\theta, p) e^{ip\lambda} (dp)$$

- Two-component evolution:

$$\frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} = -H \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} \qquad H = \begin{pmatrix} L & -1 \\ V_2 & L \end{pmatrix}$$

Here L is the Fokker-Planck operator for the Kramers problem:

$$L = p\partial_\theta + \omega^2 \sin\theta\, \partial_p - \alpha \cos^2\theta\, \partial_p^2,$$

$$V_2 = -\omega^2 \cos\theta - \alpha \sin 2\theta\, \partial_p.$$

BRST symmetry

$$Z = \int D\theta\, D\lambda\, D\overline{\chi}\, D\chi \exp\left[\int dt \left\{ i\lambda(-\ddot{\theta} + F(\theta)) + \overline{\chi}(-\partial_t^2 + F'(\theta))\chi \right\}\right]$$

- The Lagrangian can be written as

$$\mathcal{L} = \hat{\mathcal{D}}\, [\overline{\chi}(-\ddot{\theta} + F(\theta))]$$

where $\hat{\mathcal{D}}$ is a nilponent BRST operator

$$\hat{\mathcal{D}} = i\lambda\partial_{\overline{\chi}} - \chi\partial_\theta$$

- BRST symmetry of $\mathcal{L}$ translates to that of $\hat{\Psi}$:

$$\hat{\Psi} = \Psi + \Phi\overline{\chi}\chi = \hat{\mathcal{D}}\, (\overline{\chi}\psi)$$

- In terms of the superpotential ψ

$$\Psi(\theta, p) = -\partial_\theta \psi(\theta, p)$$

$$\Phi(\theta, p) = \partial_p \psi(\theta, p)$$

Superpotential: Fokker-Planch evolution

- Schrödinger equation for the superpotential

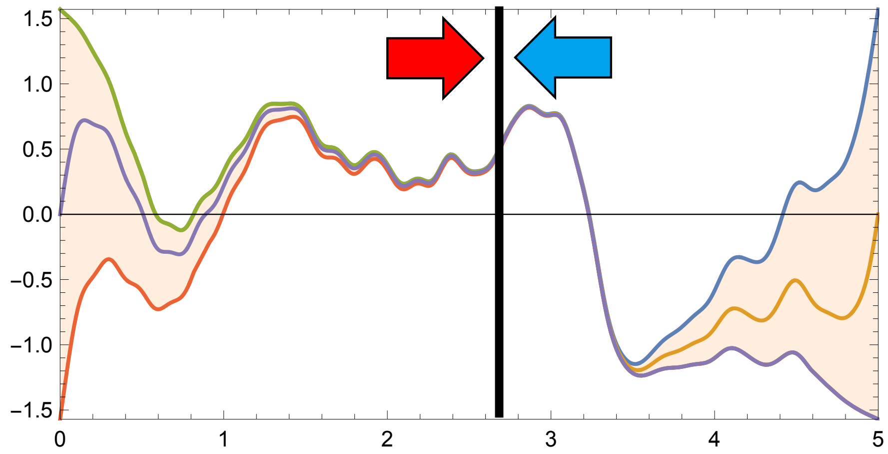
$$\frac{\partial}{\partial t}\psi(\theta,p) = -\left(p\partial_\theta + \omega^2 \sin\theta\,\partial_p - \alpha\cos^2\theta\,\partial_p^2\right)\psi(\theta,p)$$

- Mathematically equivalent to the Fokker-Planch equation for $\mathrm{PDF}(\theta,p)$. in a usual setup. We also see that $p = \partial_t\theta$.

$$\Psi(\theta,p) = -\partial_\theta\psi(\theta,p)$$
$$\Phi(\theta,p) = \partial_p\psi(\theta,p)$$

- Joint probability distribution function:

$$P(\theta,p) = \Psi(\theta,p)\Phi(\theta,-p) + \Psi(\theta,-p)\Phi(\theta,p) = \left\{\psi(\theta,p),\psi(\theta,-p)\right\}_{\theta,p}$$



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Never-falling trajectory & zero mode

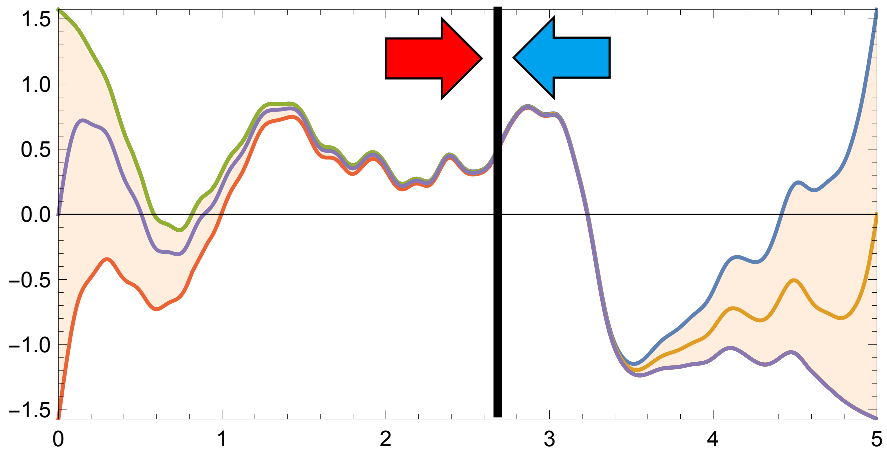
- Schrödinger equation for the superpotential

$$\frac{\partial}{\partial t}\psi(\theta,p) = -\left(p\partial_\theta + \omega^2 \sin\theta\,\partial_p - \alpha\cos^2\theta\,\partial_p^2\right)\psi(\theta,p)$$

- In the limit $T \to \infty$, only the zero mode survives:

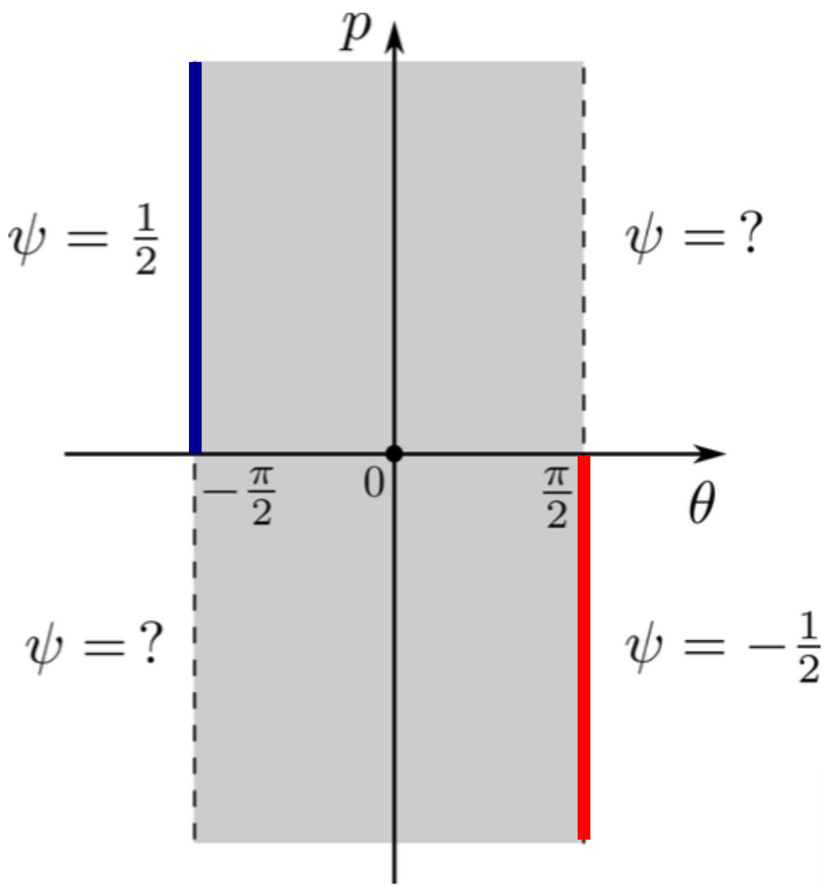
$$\boxed{\left(p\partial_\theta + \omega^2 \sin\theta\,\partial_p - \alpha\cos^2\theta\,\partial_p^2\right)\psi(\theta,p) = 0}$$

like in the theory of q1D Anderson localization



Boundary conditions

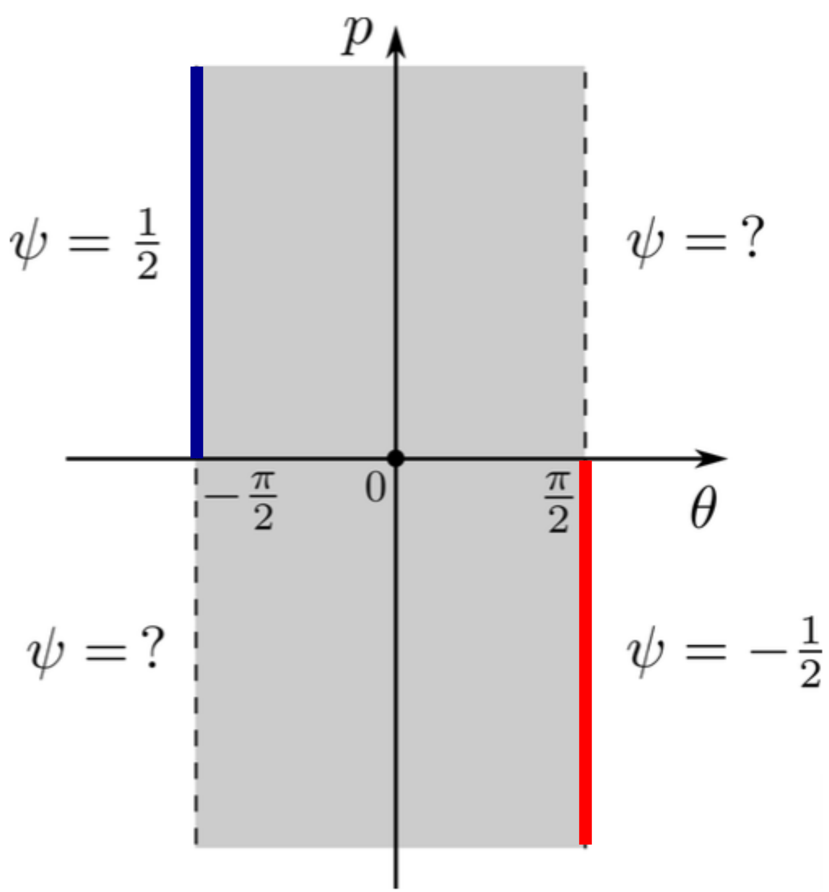
$$P(\theta, p) = \{\psi(\theta, p), \psi(\theta, -p)\}_{\theta, p}$$



Normalization condition: $\psi(\theta, p)\Big|_{p=-\infty}^{p=\infty} = 1$

Boundary conditions

$$P(\theta,p) = \{\psi(\theta,p), \psi(\theta,-p)\}_{\theta,p}$$



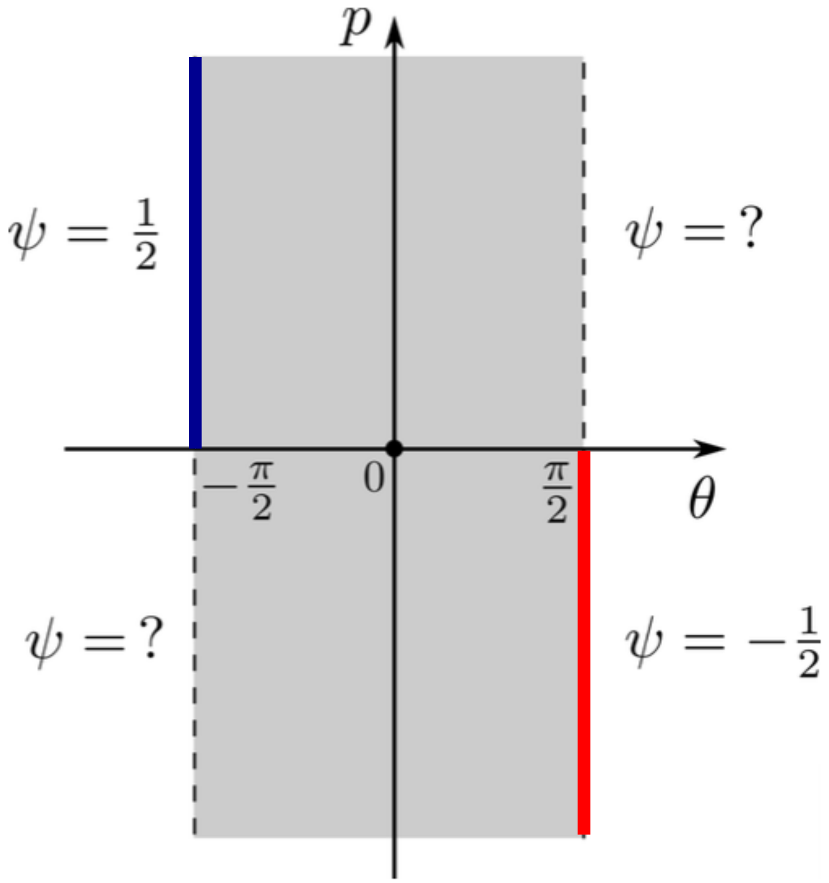
Our BC are different from the usual BC for the FPE:

- absorbing wall
- reflecting wall

They resemble BC for an absorbing wall, which however acts as a source of incoming particles with a p-independent flux

Equation to solve

$$\left(p\partial_\theta + \omega^2 \sin\theta\,\partial_p - \alpha\cos^2\theta\,\partial_p^2\right)\psi(\theta,p) = 0$$



1) No driving

- In the absence of driving ($\alpha = 0$)

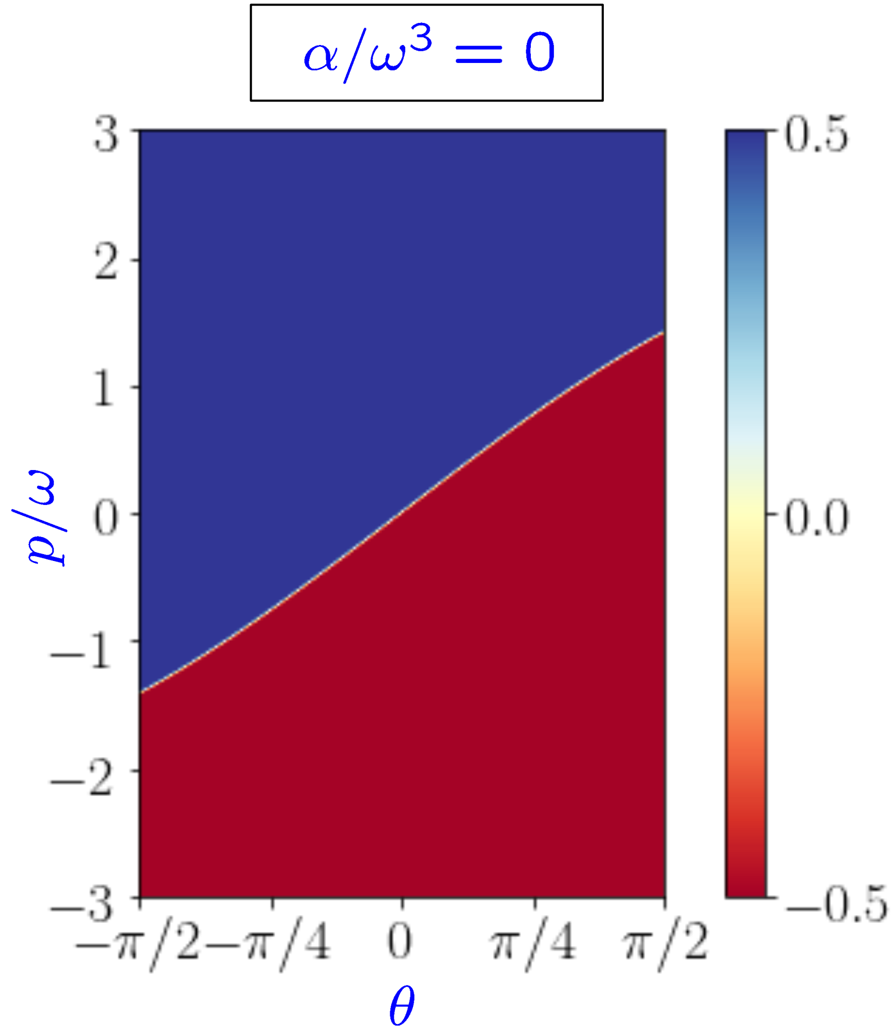
$$\left(p\partial_\theta + \omega^2 \sin\theta\, \partial_p\right) \psi(\theta, p) = 0$$

- That can be easily solved:

$$\psi(\theta, p) = \frac{1}{2}\,\mathrm{sign}(p - 2\omega \sin\theta/2)$$

- Probability distribution function:

$$P(\theta, \dot\theta) = \delta(\theta)\delta(\dot\theta)$$



2) Weak-driving limit

- If $\alpha/\omega^3 \ll 1$ then equation for ψ can be linearized ($\theta \ll 1$):

$$\left(p\partial_\theta + \omega^2\theta\partial_p - \alpha\partial_p^2\right)\psi(\theta, p) = 0$$

- That can also be solved:

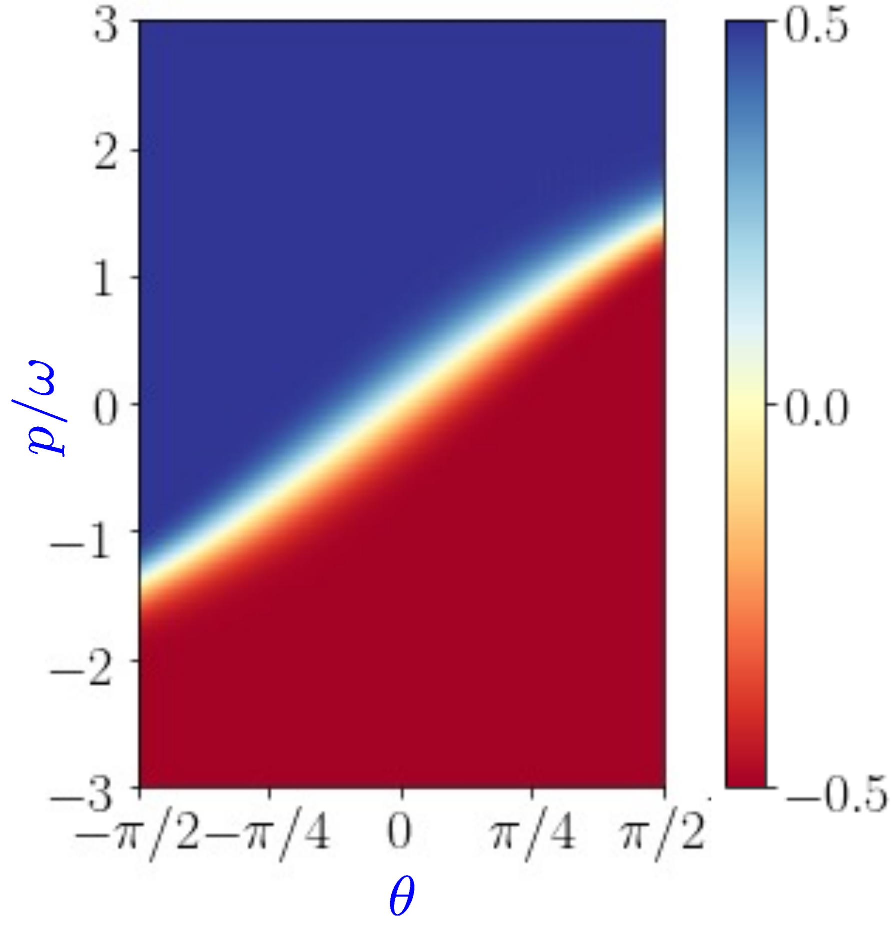
$$\psi(\theta, p) = \frac{1}{2}\,\mathrm{erf}\left[\sqrt{\frac{\omega}{2\alpha}}(p - \omega\theta)\right]$$

- Probability distribution function:

$$P(\theta, \dot{\theta}) = \frac{\omega^2}{\pi\alpha}\exp\left(-\frac{\omega^3}{\alpha}\theta^2 - \frac{\omega}{\alpha}\dot{\theta}^2\right)$$

as expected

$\alpha/\omega^3 = 0.1$



3) Strong-driving limit

- If $\alpha/\omega^3 \gg 1$ we neglect the ω-term (no gravitation limit):

$$\left(p\partial_\theta - \alpha \cos^2\theta\, \partial_p^2\right)\psi(\theta, p) = 0$$

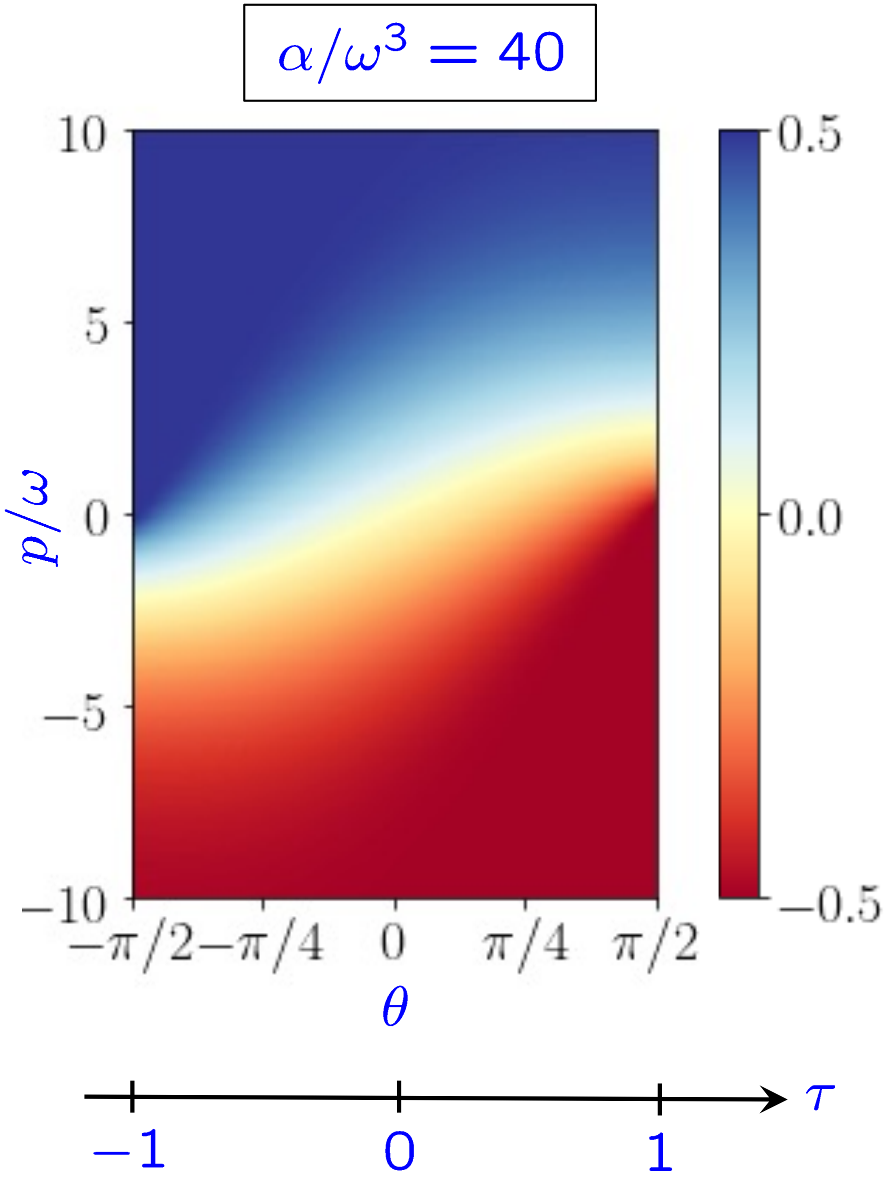
- In terms of new variables

$$\tau = \frac{4}{\pi}\int_0^\theta \cos^2\theta' d\theta' = \frac{2\theta + \sin 2\theta}{\pi}$$

$$q = (4/\pi\alpha)^{1/3} p$$

we have a pretty simple equation:

$$\partial_\tau \psi = q^{-1}\partial_q^2 \psi$$



3) Strong-driving limit

- Equation $\partial_\tau \psi = q^{-1} \partial_q^2 \psi$ is solved by the multiplicative Airy transform:

$$\psi(\tau, q) = \int_{-\infty}^{\infty} d\mu \, c(\mu) \, \mathrm{Ai}(\mu q) \exp(\mu^3 \tau)$$

- The coefficients $c(\mu)$ to be determined from the boundary conditions

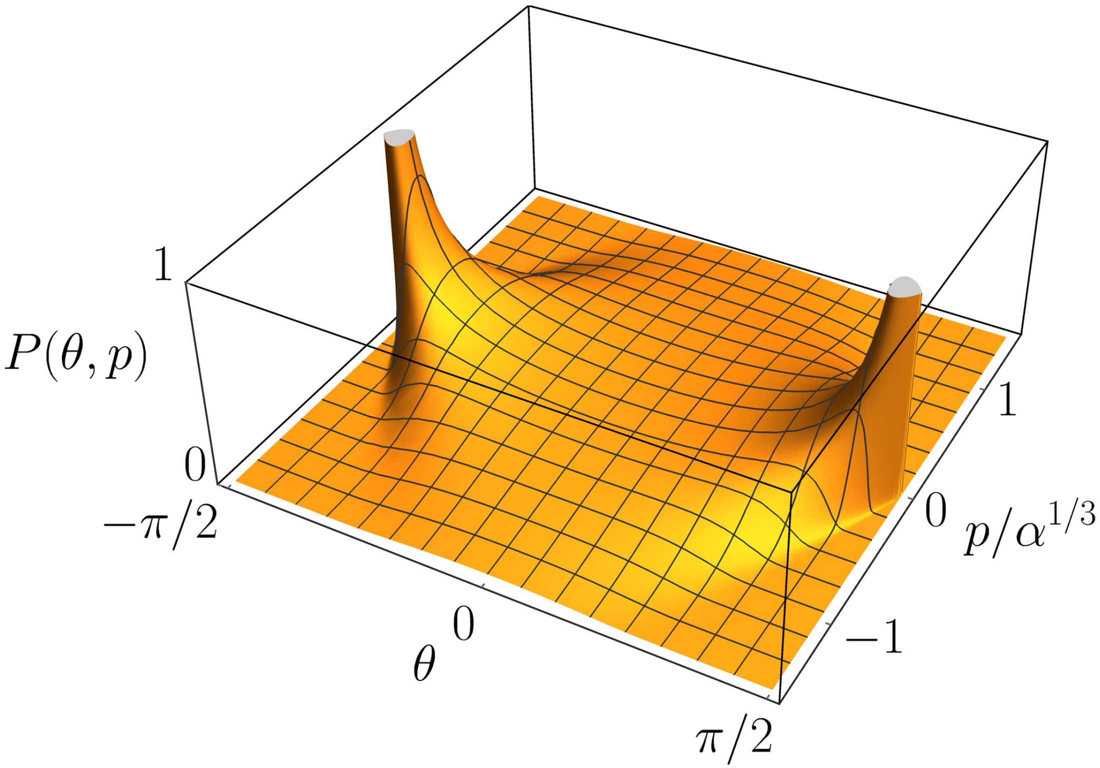
$$\psi(\tau, q) = \frac{3\,\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} \int_{-\infty}^{\infty} \frac{d\mu}{\mu} \, \mathrm{Ai}\left[(3/2)^{2/3} \mu^2\right] \mathrm{Ai}(\mu q) \exp(\mu^3 \tau)$$

$\sim \exp(-|\mu|^3)$

3) Strong-driving limit

- PDF $P(\theta,p)$ $[s=(2/3\pi\alpha)^{1/3}p/(1-\tau^2)^{1/3}]$:

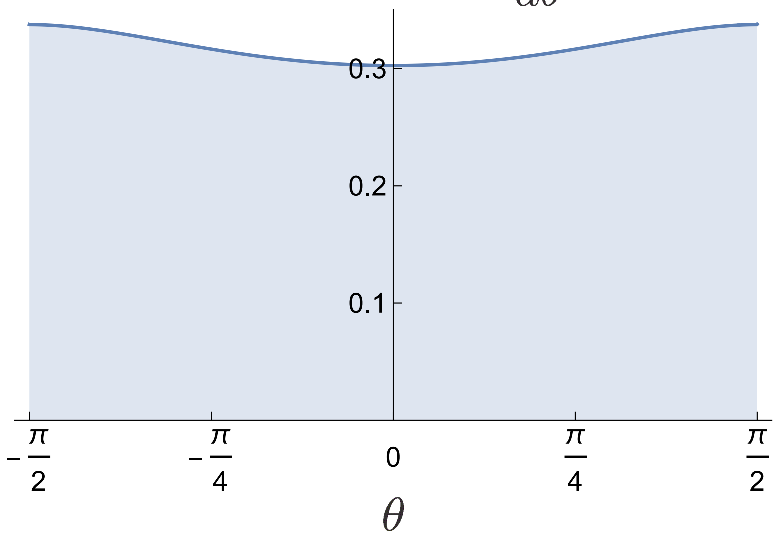
$$P(\theta,p) = -\frac{16\cos^2\theta}{3^{1/3}\pi^{4/3}\alpha^{1/3}}\left(\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)}\right)^2\frac{\mathrm{Ai}(s^2)\,\mathrm{Ai}'(s^2)}{1-\tau^2} \qquad \tau = \frac{2\theta+\sin 2\theta}{\pi}$$



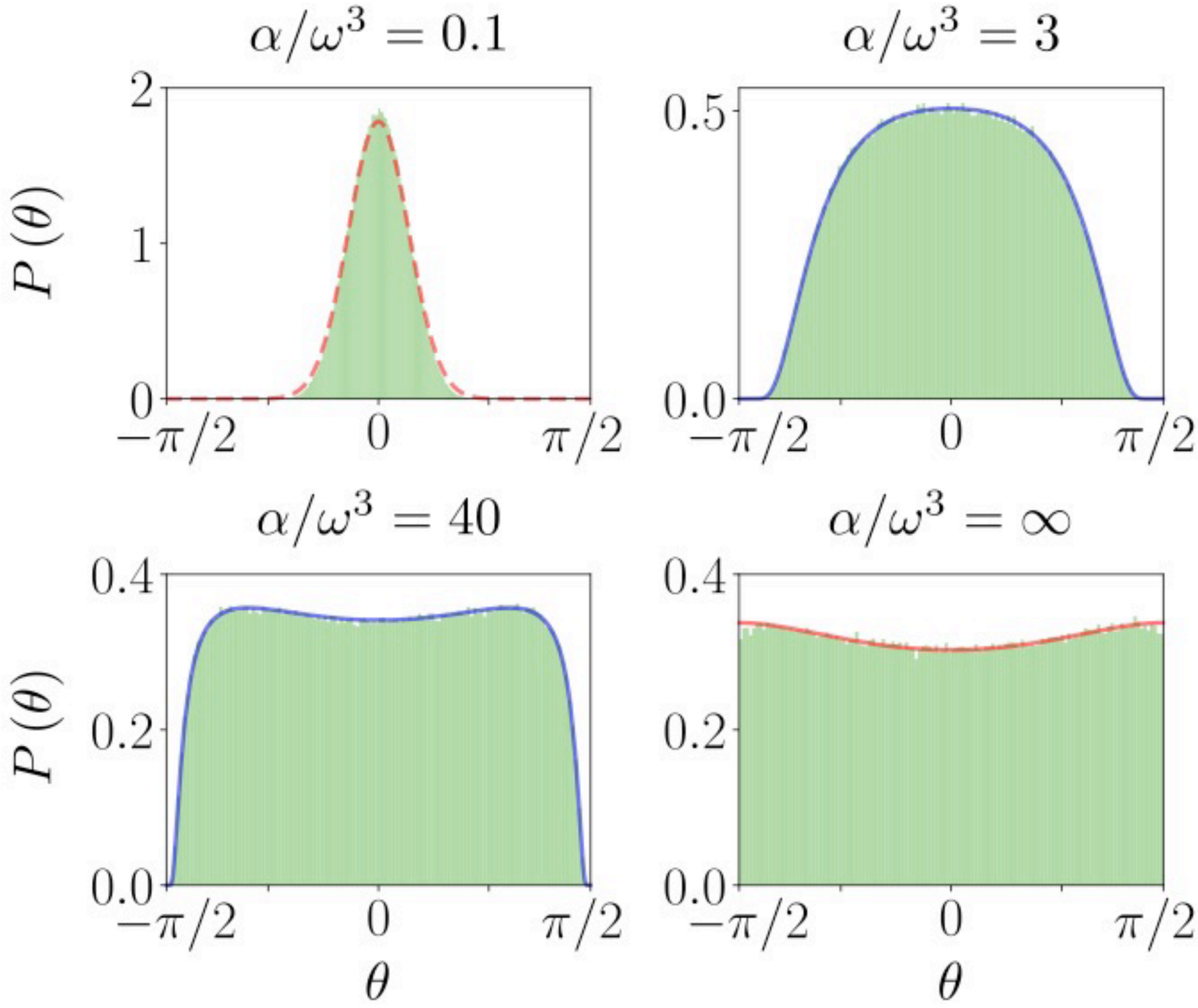
- $P(\theta)$ distribution function:

$$P(\tau) = \frac{\Gamma(5/6)}{\Gamma(1/3)\Gamma(1/2)}\frac{1}{(1-\tau^2)^{2/3}}$$

$$P(\theta) = P(\tau)\frac{d\tau}{d\theta}$$



Comparison to direct Monte-Carlo simulation



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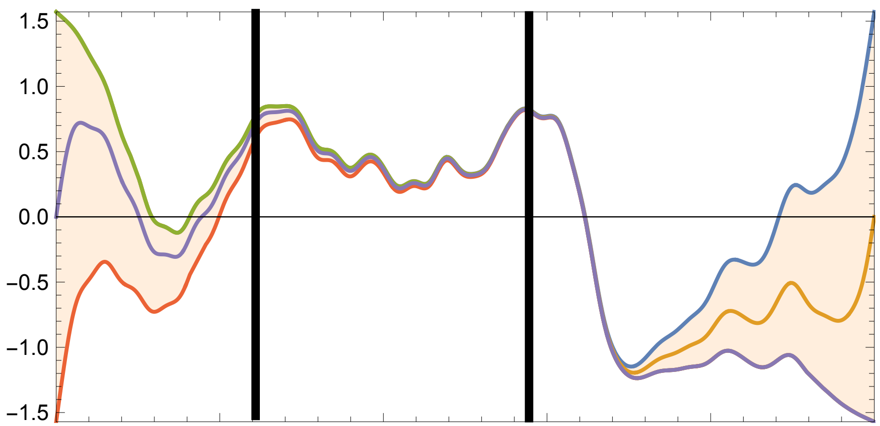
Lyapunov exponent

- Schrödinger equation for the superpotential

$$\frac{\partial}{\partial t}\psi(\theta,p) = -\left(p\partial_\theta + \omega^2 \sin\theta\,\partial_p - \alpha\cos^2\theta\,\partial_p^2\right)\psi(\theta,p)$$

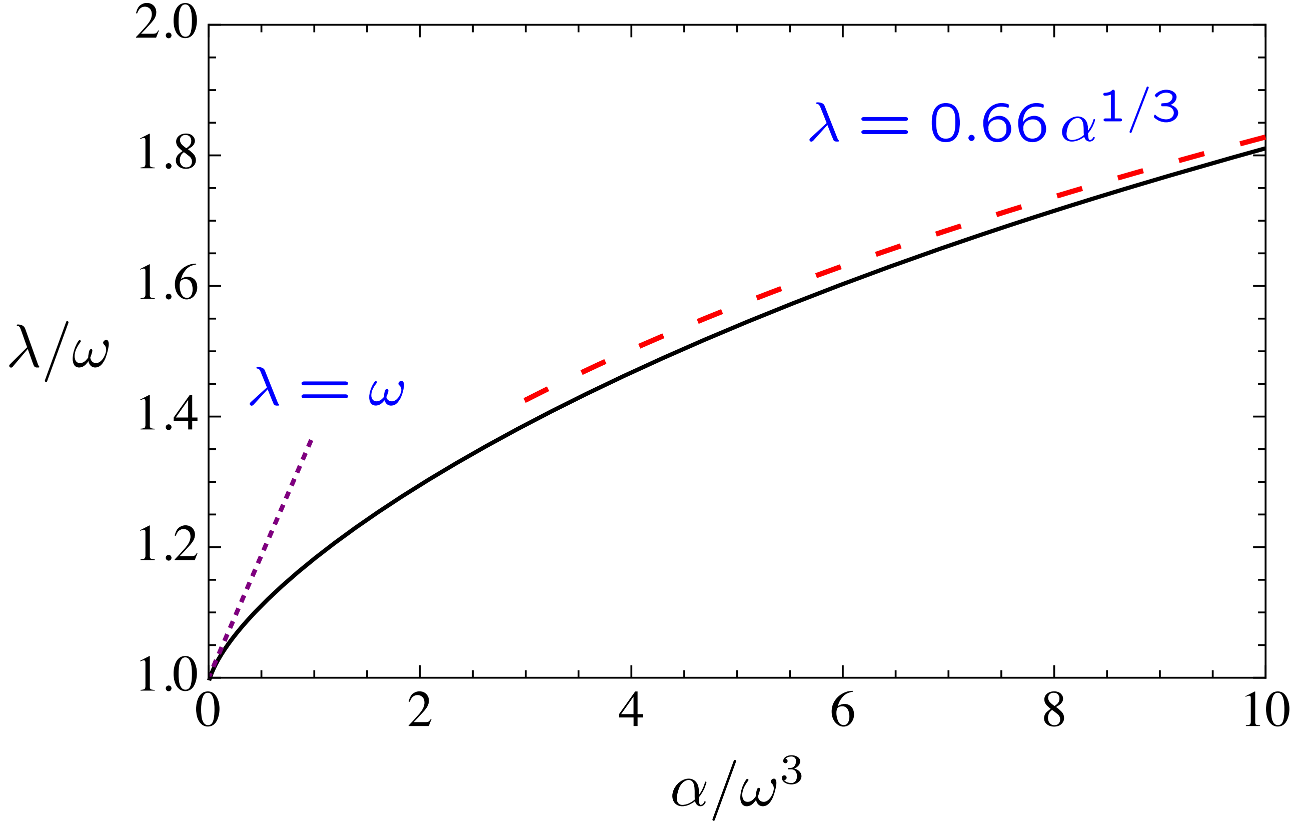
- While *one-time* statistics of the *never-falling* trajectory is expressed by the zero mode, *different-time* statistics or one-time statistics *at finite intervals* are determined by the spectrum of the FP operator.

- Lyapunov exponent: $\lambda = -\lim_{t\to\infty}\frac{\partial \ln\langle\theta(0)\theta(t)\rangle}{\partial t}$



Lyapunov exponent

- Lyapunov exponent: $\lambda = -\lim_{t\to\infty} \frac{\partial \ln\langle\theta(0)\theta(t)\rangle}{\partial t} = \omega g(\alpha/\omega^3)$



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Physical meaning of SUSY objects

- SUSY wave function

$$\hat{\Psi} = \Psi + \Phi\bar{\chi}\chi \qquad \Psi(\theta,p) = -\partial_\theta\psi(\theta,p) \quad \Phi(\theta,p) = \partial_p\psi(\theta,p)$$

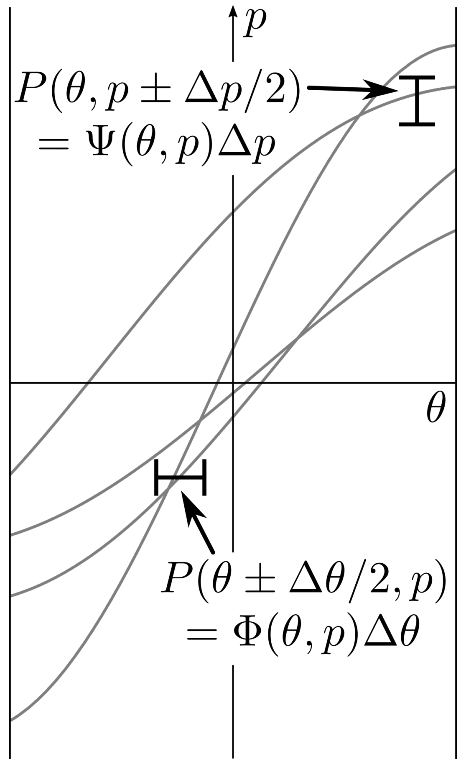
- Velocity profile $p(\theta,t)$

- FP equation for $\psi = \langle\Theta\left(p(\theta,t) - p\right)\rangle$

$$\frac{\partial}{\partial t}\psi(\theta,p) = -\left(p\partial_\theta + \omega^2\sin\theta\,\partial_p - \alpha\cos^2\theta\,\partial_p^2\right)\psi(\theta,p)$$

- In terms of the profile $p(\theta,t)$, Ψ and Φ give conditional probability to have p at a given θ and vise versa:

$$\Psi\left(\theta,p\right) = P\left(p|\theta,t\right)$$
$$\Phi\left(\theta,p\right) = P\left(\theta|p,t\right)$$

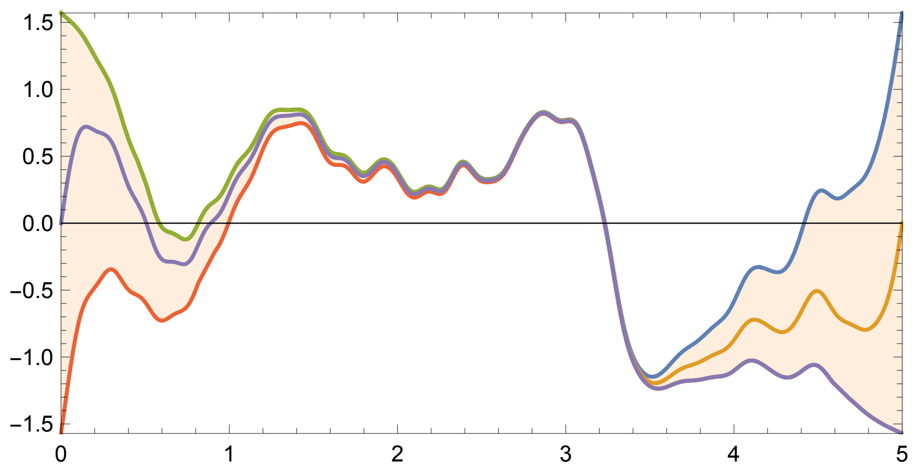


Outlook

- **1D inhomogeneous superconductors**
  Usadel equation as pendulum equation with complex ω
  hope for exact determination of the DOS peak smearing

- **Sinai-Khanin minimizers**
  E, Khanin, Mazel & Sinai (2000), Khanin et al. (2000—)
  Trajectory that provides true minimum for the action, not a saddle point

- **Randomly forced Burgers equation** (1D pressureless turbulence)
  Polyakov (1995), Gurarie & Migdal (1996), . . .
  Equation for characteristics is in the class of the driven pendulum equation



Summary

- Statistical properties of a unique non-falling trajectory have been analyzed
- Supersymmetric Parisi-Sourlas approach + transfer-matrix method
- The PDF $P(\theta, \dot{\theta})$ is obtained as a "square" of an auxiliary superpotential, which obeys the Fokker-Planch equation with a proper BCs
- Never-falling trajectoty = zero mode
  Finite-time correlations = excited states
- Can be derived without supersymmetry

N. A. Stepanov and M. A. Skvortsov, SciPost Phys. **13**, 021 (2022)

N. A. Stepanov and M. A. Skvortsov, JETP Lett. **112**, 376 (2020)