QUANTUM BUTTERFLY EFFECT WITHOUT FALSE POSITIVES

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Abstract

Out-of-time order correlators are widely used as a measure of quantum chaos and quantum butterfly effect, but give false-positive quantum Lyapunov exponents in integrable systems with unstable fixed points. I suggest an alternative measure of quantum chaos, which does not have this problem. To illustrate the approach, I calculate true quantum Lyapunov exponents numerically in the Lipkin-Meshkov-Glick and Feingold-Peres models and analytically in the large-N vector mechanics.

1. False signatures of quantum chaos in OTOCs

- Quantum chaos and quantum butterfly effect are frequently defined using the **out-of-time-order correlation functions** (OTOCs):

$$\mathrm{OTOC}(t) = \frac{1}{N\hbar^2}\sum_{i,j}\left\langle [\hat{z}_i(t), \hat{z}_j(0)]^\dagger [\hat{z}_i(t), \hat{z}_j(0)] \right\rangle \sim \left\langle e^{2\kappa_{cl} t} \right\rangle \sim e^{2\kappa_q t}, \qquad (1)$$

where $\langle \cdots \rangle$ denotes the averaging over a thermal ensemble, N is the phase space dimension, $\mathbf{z} = (\mathbf{q}, \mathbf{p})$ are canonical coordinates, κ_{cl} is the largest Lyapunov exponent (LE), and κ_q is the **(naive) quantum LE**.
- Namely, one usually defines quantum chaos through $\kappa_q > 0$
- Classical chaos is defined through $\bar{\kappa}_{cl} = \langle \kappa_{cl} \rangle > 0$ (Kolmogorov system)
- **Definition $\kappa_q > 0$ does not reproduce $\bar{\kappa}_{cl} > 0$ in the semiclassical limit!**

2. The LOTOC and the true quantum LE

- To close this loophole, we suggest an alternative measure of quantum chaos and quantum butterfly effect — the **logarithmic OTOC** (LOTOC):

$$C(t) = \left\langle \log\left(\frac{1}{N\hbar^2}\sum_{i,j} [\hat{z}_i(t), \hat{z}_j(0)]^\dagger [\hat{z}_i(t), \hat{z}_j(0)] \right) \right\rangle, \qquad (2)$$

- The **true quantum LE** $\bar{\kappa}_q$ is extracted from the linear growth of the LOTOC up to the Ehrenfest time:

$$C(t) \approx 2\bar{\kappa}_q t + o(t), \quad 1 \ll t \ll t_E, \qquad (3)$$

where $o(t)$ grows slower than linearly (e.g., $o(t) \sim \log t$).
- We argue that $\bar{\kappa}_q \to \bar{\kappa}_{cl}$ as $\hbar \to 0$.
- In other words, **the LOTOC suggests a proper definition of the quantum butterfly effect — the exponential sensitivity to *almost all* small perturbations ensured by a positive true qLE, $\bar{\kappa}_{cl} > 0$**

3. Lipkin-Meshkov-Glick model

- As an illustrative example of an **integrable system** with an isolated saddle point, we consider the Lipkin-Meshkov-Glick model:

$$\hat{H}_{\mathrm{LMG}} = \hat{x} + 2\hat{z}^2, \qquad (4)$$

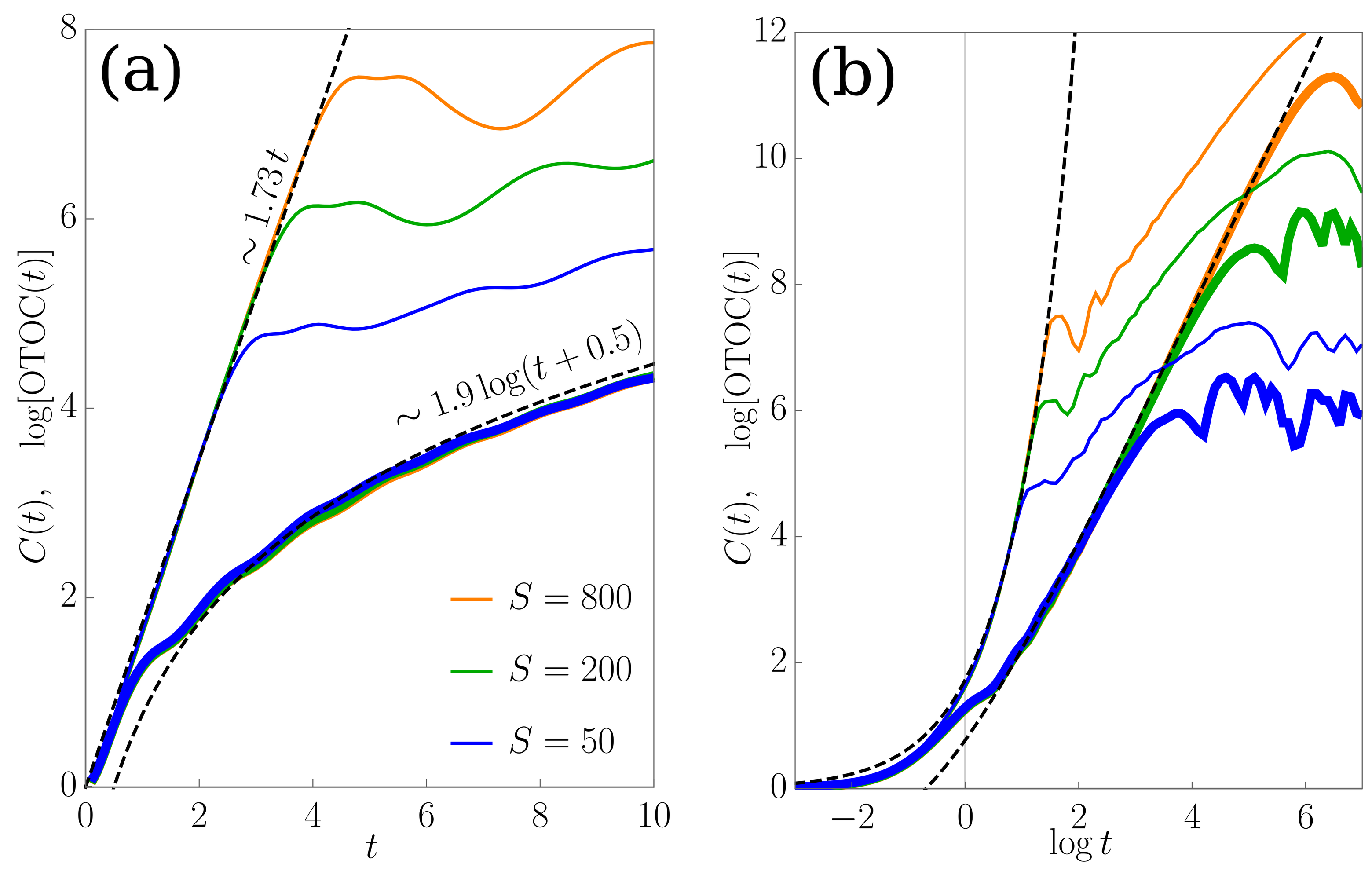
where $\hat{x}, \hat{z} = \hat{S}_x/S, \hat{S}_z/S$ are rescaled $SU(2)$ spin operators with total spin S.
- In the classical limit $S \to \infty$, this model has an isolated saddle point $x = 1$, where $\partial z_i(t; \mathbf{z}_0)/\partial z_{0j} \sim e^{\kappa_s t}$ with $\kappa_s = \sqrt{3}$.
- The **OTOC grows exponentially** up to the "chaotic" Ehrenfest time,

$$\mathrm{OTOC}(t) \sim e^{2\kappa_q t} \quad \text{for} \quad 1 \lesssim t \lesssim \log(1/\hbar). \qquad (5)$$

- The **LOTOC grows logarithmically** until the "integrable" Ehrenfest time:

$$C(t) \sim \log t \quad \text{for} \quad 1 \lesssim t \lesssim 1/\hbar \qquad (6)$$

- From numerics, we estimate $\kappa_q = \kappa_s/2$ and $\bar{\kappa}_q = \bar{\kappa}_{cl} = 0$



4. Feingold-Peres model

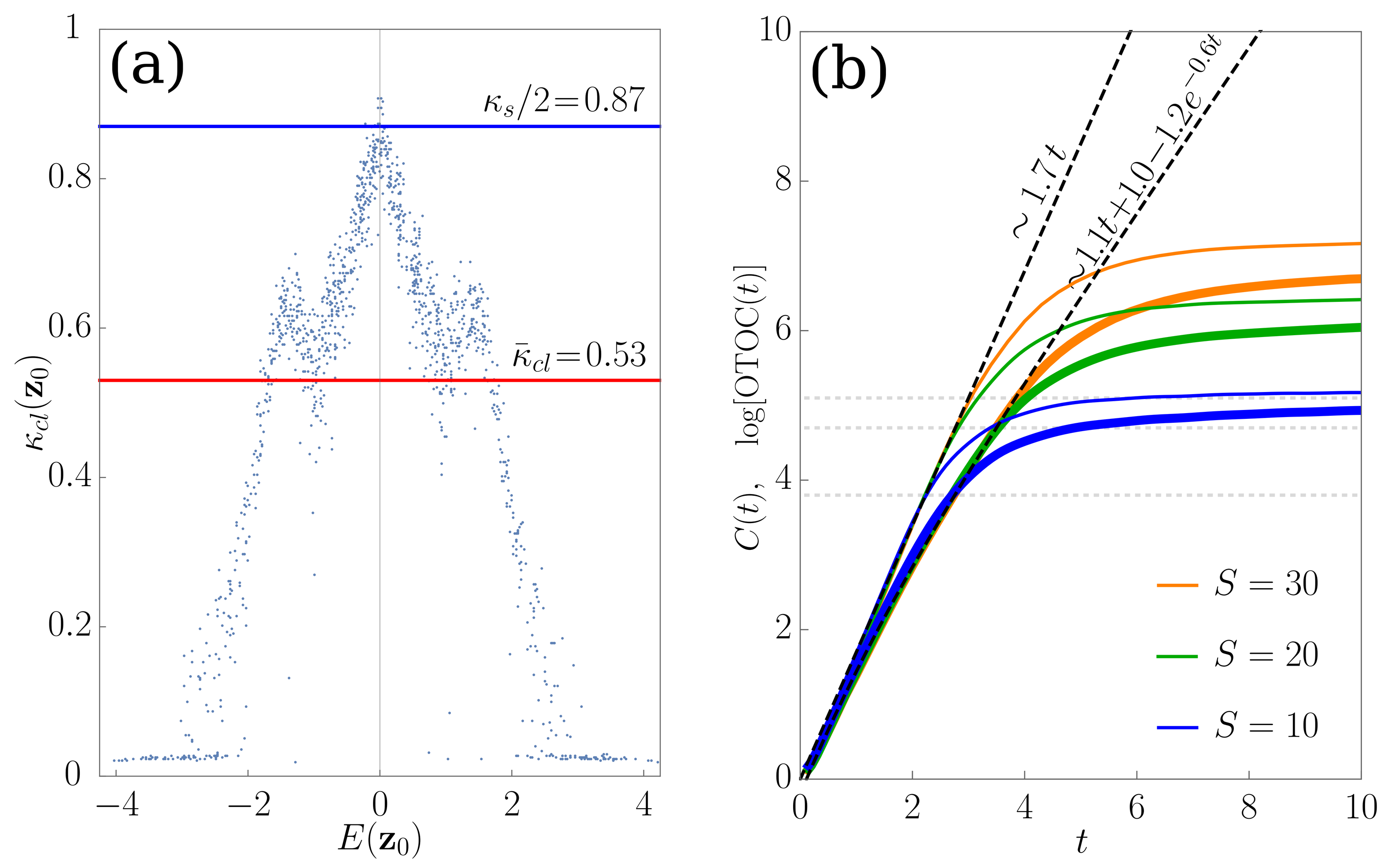
- To study the behavior of the OTOC and LOTOC in a **truly chaotic system**, we consider the Feingold-Peres (FP) model:

$$\hat{H}_{\mathrm{FP}} = \hat{x}_1 + \hat{x}_2 + 4\hat{z}_1\hat{z}_2, \qquad (7)$$

where $(\hat{x}_i, \hat{y}_i, \hat{z}_i)$ are two independent rescaled $SU(2)$ spin operators.
- In the semiclassical limit $S \to \infty$, FP model has positive LEs for the majority of initial conditions, so $\bar{\kappa}_{cl} \approx 0.53$, see figure (a).
- It also has two isolated saddle points $x_1 = x_2 = \pm 1$, where $\partial z_i(t; \mathbf{z}_0)/\partial z_{0j} \sim e^{\sqrt{3}t}$
- In FP model, **both OTOC and LOTOC grow chaotically** until the "chaotic" Ehrenfest time:

$$\mathrm{OTOC}(t) \sim e^{2\kappa_q t} \quad \text{and} \quad C(t) \approx 2\bar{\kappa}_q t \quad \text{for} \quad 1 \lesssim t \lesssim \log(1/\hbar). \qquad (8)$$

- From numerics, we again estimate $\kappa_q = \kappa_s/2$ and $\bar{\kappa}_q = \bar{\kappa}_{cl} > 0$



5. Nonlinear vector mechanics

- Now let us consider the system of $N \gg 1$ nonlinearly coupled oscillators with an explicitly broken $O(N)$ symmetry:

$$\hat{H} = \frac{1}{2}\hat{p}_i^2 + \frac{1}{2}m^2\hat{x}_i^2 + \frac{\lambda}{4N}\hat{x}_i^2\hat{x}_j^2 - \frac{\lambda}{4N}\hat{x}_i^4. \qquad (9)$$

- We estimate the LOTOC and the true quantum LE using the **replica trick**:

$$C(t) = \lim_{n\to 0} \frac{\partial C_n(t)}{\partial n}, \qquad (10)$$

where we introduce the replica OTOC (ROTOC):

$$C_n(t) = \left\langle \left(\frac{1}{N\hbar^2}\sum_{i,j} [\hat{z}_i(t), \hat{z}_j(0)]^\dagger [\hat{z}_i(t), \hat{z}_j(0)] \right)^n \right\rangle. \qquad (11)$$

- Substituting the exponentially growing ansatz $C_n(t) \sim e^{2\kappa_n t}$ to the Dyson-Schwinger equation on the resummed ROTOC, we estimate the replica LE:

$$\kappa_n = n\left[(2n-1)!!\right]^{\frac{1}{2n}} \frac{8\sqrt{6}}{N} \frac{\lambda\tilde{m}}{(\tilde{\mu}\tilde{m})^3} \frac{e^{\tilde{\beta}\tilde{m}/2}}{e^{\tilde{\beta}\tilde{m}} - 1}, \qquad (12)$$

- The true quantum LE, $\bar{\kappa}_q \approx 0.7\sqrt[4]{\lambda T}/N$, found from Eq. (10), is approximately two times smaller than the naive one, $\kappa_q \approx 1.3\sqrt[4]{\lambda T}/N$.
- From the semiclassical perspective, **this discrepancy arises because the LOTOC measures the average of LEs over the entire phase-space, whereas the OTOC singles out only the points with the largest LEs**.