Theoretical description of coherent geostrophic vortices and comparison with experiment

Sergey Vergeles

with

Parfenyev Vladimir, Igor Kolokolov, Leon Ogorodnikov, Ivan Vointsev,

Alexander Levchenko, Sergey Filatov, Daniil Tumachev

Landau Institute for Theoretical Physics

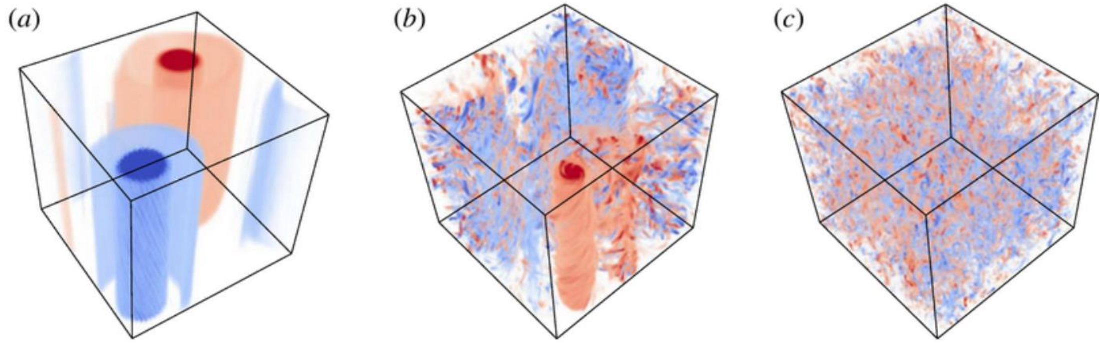
Institute of Solid State Physics

Under support of RScF grant No. 23-72-30006

Flow of fluid which fast rotates as a whole

Navier-Stokes equation for rotating flouid

(Seshasayanan & Alexakis, JFM 2018)



Navier-Stokes equation in rotating frame (angulat velocity $\boldsymbol{\Omega} = \mathbf{e}_z \Omega$)

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\nabla)\boldsymbol{v} + 2[\boldsymbol{\Omega} \times \boldsymbol{v}] = -\frac{1}{\rho}\nabla P + \nu \Delta \boldsymbol{v} \quad (1)$$

$2[\boldsymbol{\Omega} \times \boldsymbol{v}]$ is Coriolis force. Vorticity equation $\boldsymbol{\varpi} = \mathrm{rot}\, \boldsymbol{v}$

$$\partial_t \boldsymbol{\varpi} + \Big((\boldsymbol{v}\cdot\nabla)\boldsymbol{\varpi} - (\boldsymbol{\varpi}\cdot\nabla)\boldsymbol{v}\Big) - 2\Omega\partial_z \boldsymbol{v} = \nu\Delta\boldsymbol{\varpi} \quad (2)$$

If the rotation is fast as compared to the rate of the velocity alternation, then the velocity field is almost uniform along the rotation axis,

$$|\partial_z \boldsymbol{v}| \sim \frac{\partial_t}{\Omega}|\boldsymbol{\varpi}| \ll |\boldsymbol{\varpi}|$$

This is Taylor-Proudman theorem.

Geostrophic flow

Flow which is uniform along axis of rotation

$$\partial_t U^\alpha - 2\Omega\epsilon^{\alpha\beta}U^\beta + (\boldsymbol{U}\cdot\nabla)U^\alpha \;=\; -\partial_\alpha p + \nu\Delta U^\alpha$$

For the incompressible flow, planar velocity is parametrized by one scalar function, $U^\alpha = \epsilon^{\alpha\beta}\partial_\beta\Psi$ ($\Psi(t,x,y)$ — current function). Coriolis force is pure potential,

$$-\partial_\alpha p + 2\Omega\epsilon^{\alpha\beta}v^\beta = -\partial_\alpha\big(p + 2\Omega\Psi\big) \equiv -\partial_\alpha\tilde{p}.$$

If Rossby number Ro is low,

$$\mathrm{Ro} \sim \frac{|\nabla\boldsymbol{U}|}{2\Omega} \ll 1,$$

then the residue variation of pressure $\tilde{p}$ is weak as compared to the geostrophic part, $\tilde{p} \ll p$. Equation $p \approx -2\Omega\Psi$ is called geotrophic balance. The equation governing the geostrophic flow dynamics does not contain rotation:

$$\partial_t U^\alpha + (\boldsymbol{U}\cdot\nabla)U^\alpha \;=\; -\partial_\alpha\tilde{p} + \nu\Delta U^\alpha.$$

Inertial waves

Dynamics of inertial waves in linear approximation

$$\partial_t \boldsymbol{u} + 2[\boldsymbol{\Omega} \times \boldsymbol{u}] = -\nabla p + \nu \Delta \boldsymbol{u}.$$

Two circular polarizations

$$\boldsymbol{u}_{\mathbf{k}} = \sum_{s=\pm 1} a_{\mathbf{k}s} \boldsymbol{h}_{\mathbf{k}}^s, \qquad \boldsymbol{h}_{\mathbf{k}}^s = \frac{[\mathbf{k} \times [\mathbf{k} \times \boldsymbol{e}_z]] + isk\,[\mathbf{k} \times \boldsymbol{e}_z]}{\sqrt{2}\, k k_\perp}.$$

$a_{-\mathbf{k}s} = a_{\mathbf{k}s}^*$, orthogonality $(\boldsymbol{h}_{\mathbf{k}}^{-\sigma}, \boldsymbol{h}_{\mathbf{k}}^s) = \delta^{\sigma s}$.

Oscillation equation:

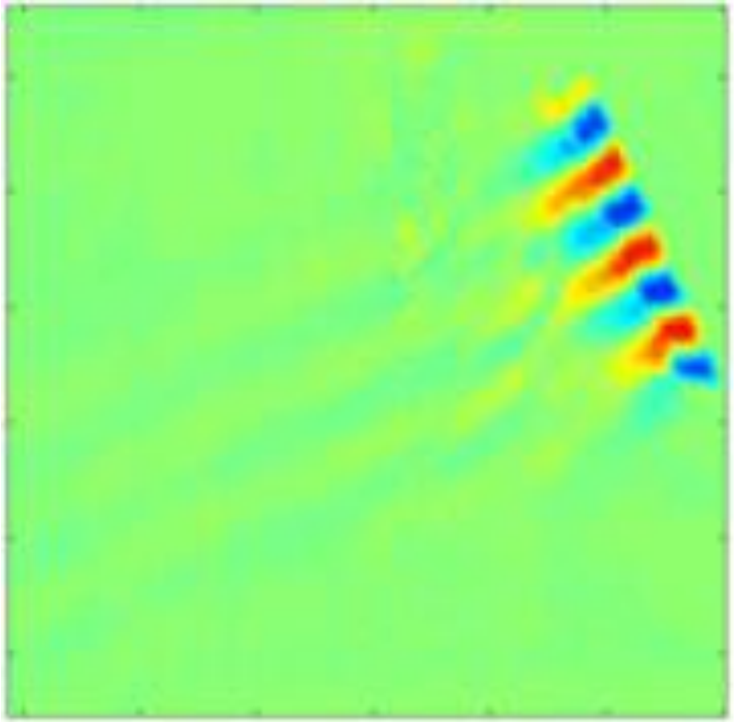
$$\partial_t a_{\mathbf{k}s} = -is\omega_{\mathbf{k}} a_{\mathbf{k}s} - \nu \mathbf{k}^2 a_{\mathbf{k}s}, \qquad \omega_{\mathbf{k}} = 2\Omega k_z / k = 2\Omega \cos\theta_{\mathbf{k}}.$$

Kinetic energy

$$E = \int \frac{\boldsymbol{u}^2}{2} \mathrm{d}^3 r = \sum_{s=\pm 1} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \frac{|a_{\mathbf{k}s}|^2}{2},$$

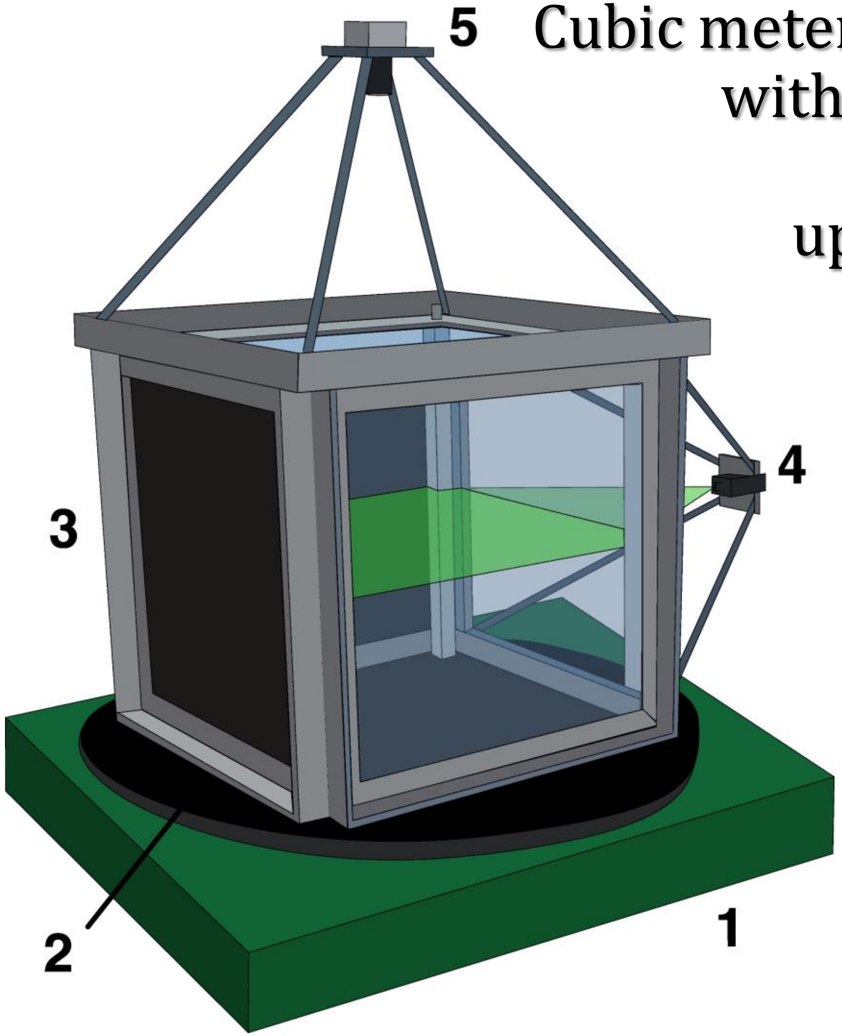
Group velocity lies in plane of $\mathbf{k}$, $\boldsymbol{\Omega}$

$$\mathbf{v}_{\mathrm{g}} = \frac{2s\Omega}{k}\big(\mathbf{e}_z - (\boldsymbol{n}_{\mathbf{k}} \cdot \mathbf{e}_z)\boldsymbol{n}_{\mathbf{k}}\big), \qquad \mathbf{v}_{\mathrm{g}} \perp \mathbf{k}, \qquad |\mathbf{v}_{\mathrm{g}}| = \frac{2\Omega \sin\theta_{\mathbf{k}}}{k}.$$

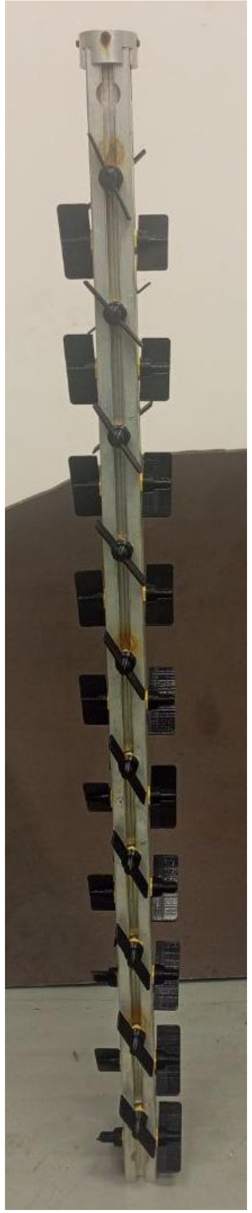


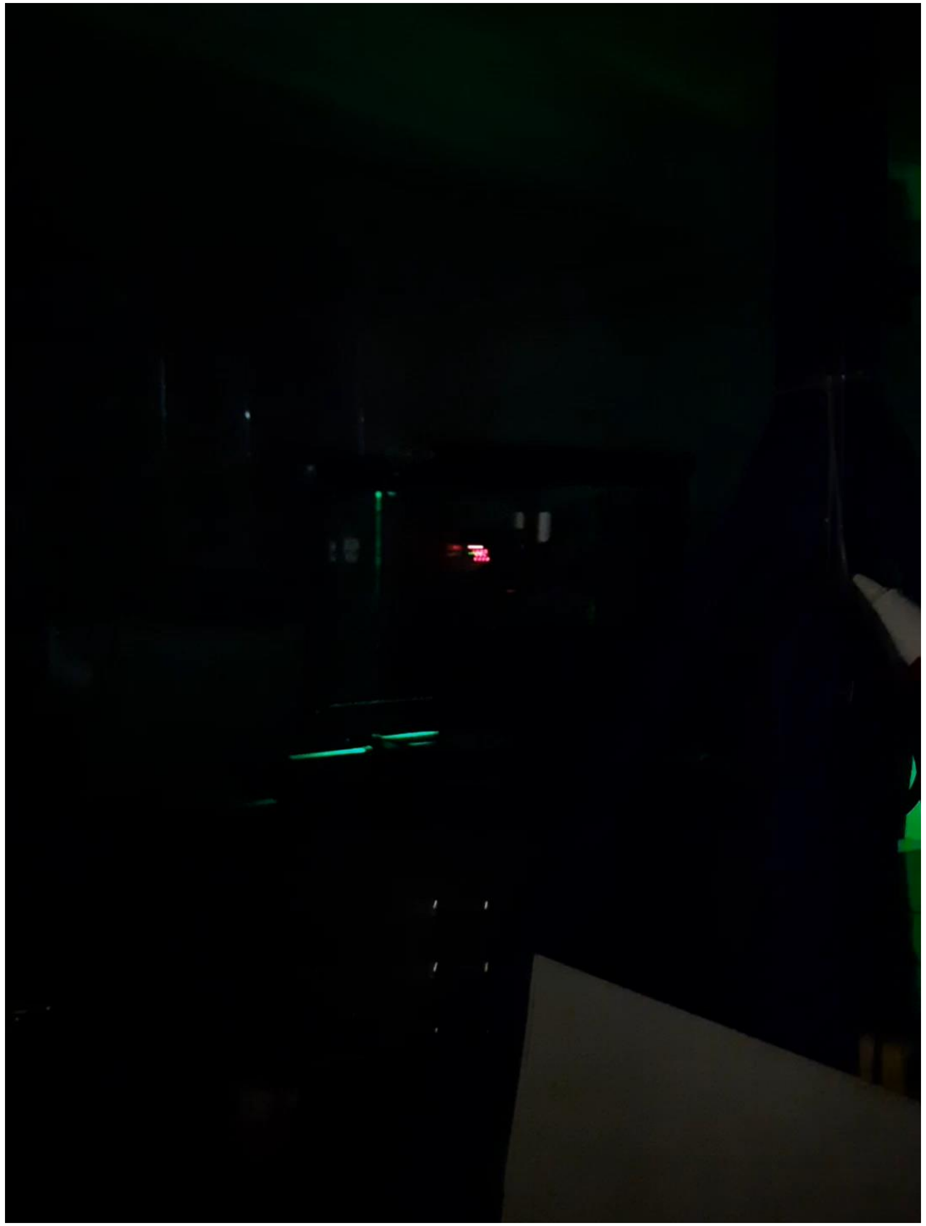
Guilhem et.al., PoF 24 p.014105 (2012)

Experimental setup



Cubic meter rotates with angular velocity up to 1 Hz





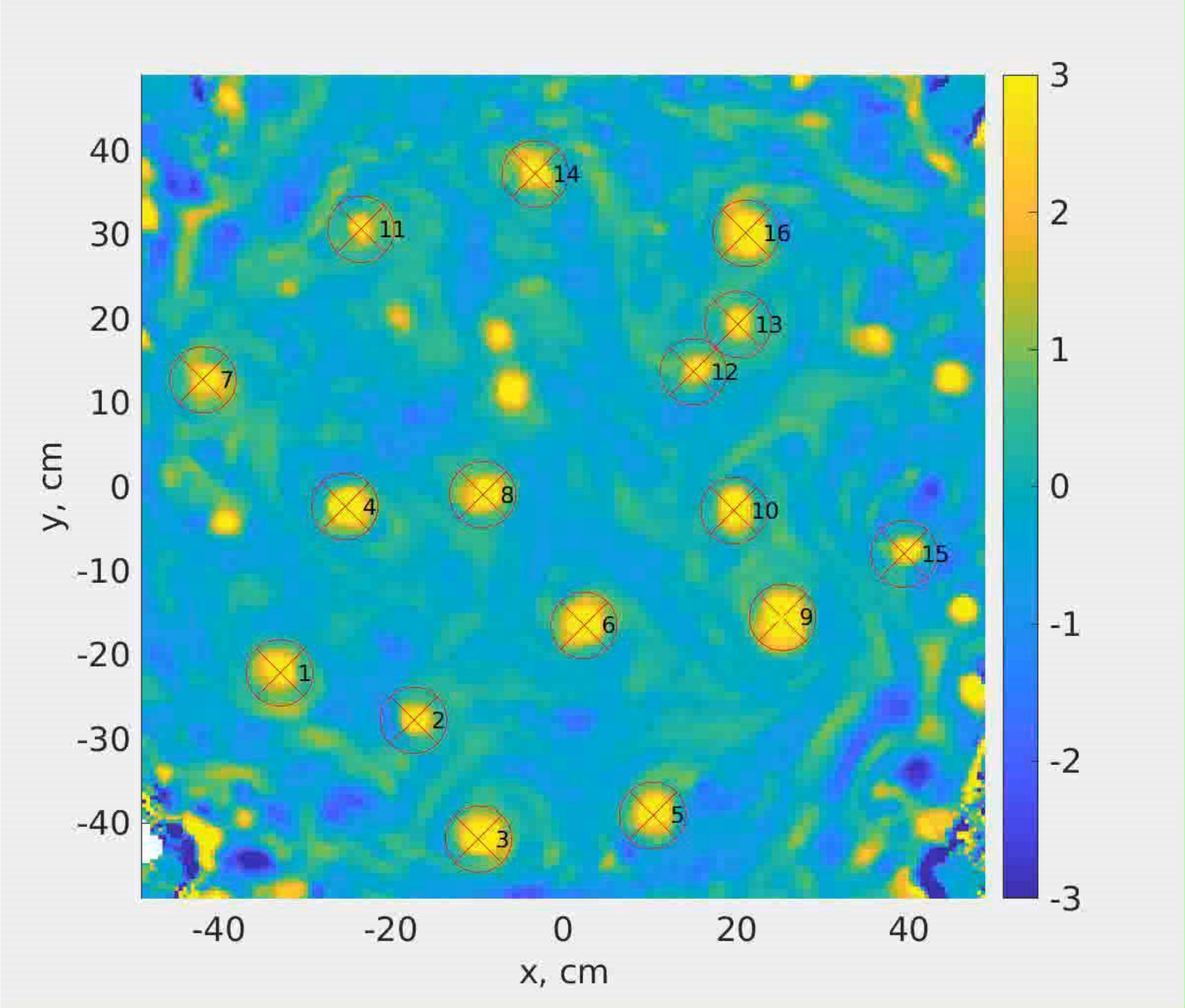
Experimental observation of column vortices (Taylor columns)

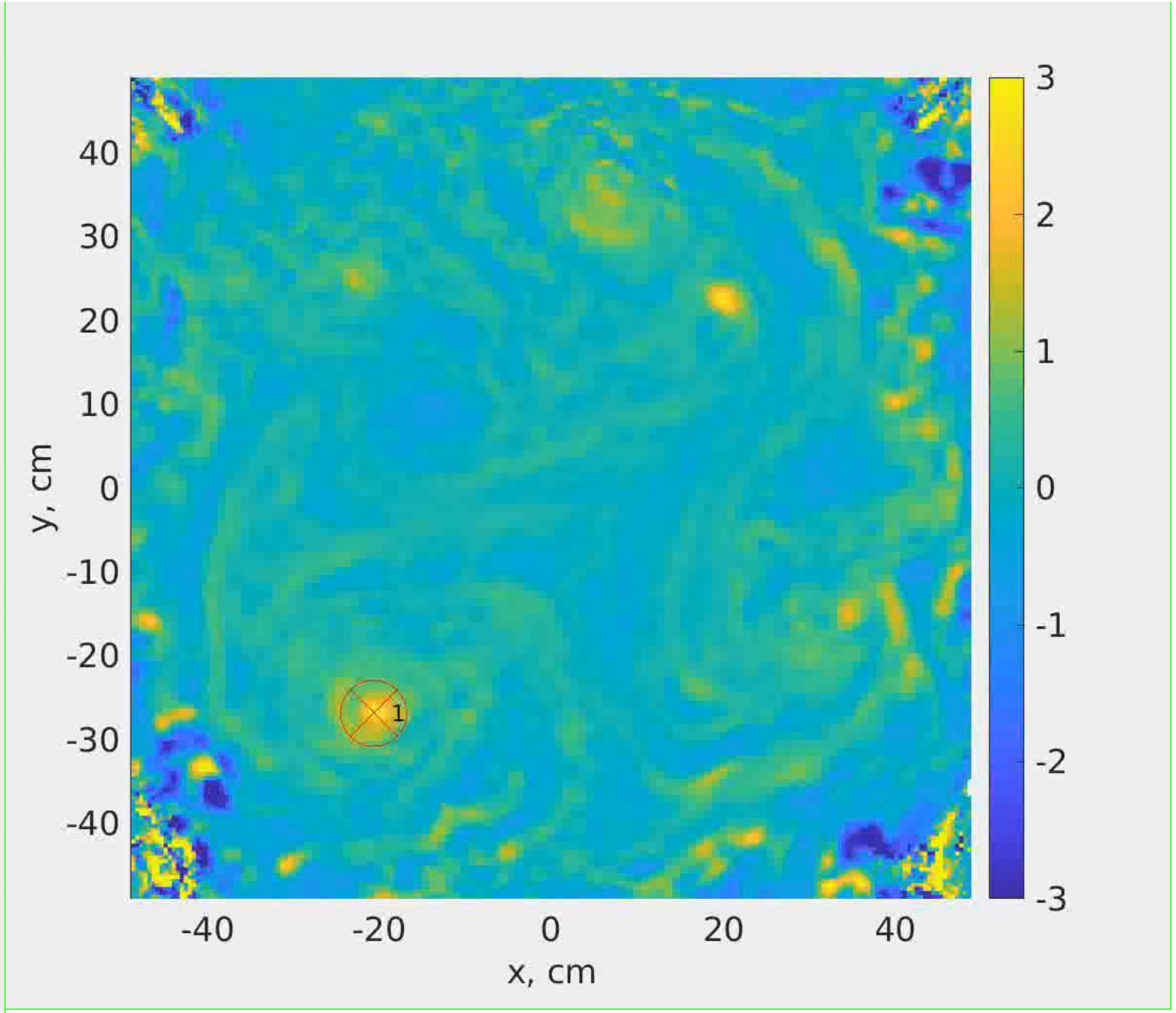
- Cyclone radial profile
  - McEwan, A. D. (1976) Nature;
  - Hopfinger, E. J., Browand, F. K., & Gagne, Y. (1982). JFM.
- The velocity field division onto quasi-2D and 3D flows
  - Ruppert-Felsot, Praud, Sharon & Swinney (2005) PRE;
  - Campagne, Gallet, Moisy & Cortet (2015) PRE
- Cyclone-anticyclone asymmetry
  - Gallet, Campagne, Cortet & Moisy (2014) PoF
  - Boffetta, Toselli, Manfrin, & Musacchio (2021). J.Turbulence **(decaying turbulence)**
- Formation of Taylor columns, interaction of quasi-2D and 3D sectors of flow
  - Brunet, Gallet, & Cortet (2020) PRL
  - Brons, Thomas, & Pothérat (2020) JFM
- Review of recent experiments
  - Godeferd, & Moisy (2015) Appl.Mech.Rev.

Rotation 0.72 Hz

$$\omega = \partial_x U^y - \partial_y U^x$$

Rotation 0.125 Hz





Geostrophic flow in a closed volume

Flow of a rotating incompressible liquid inside a vessel of hight H:

$$\partial_t \boldsymbol{v} + 2\,[\boldsymbol{\Omega}\times\boldsymbol{v}] + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \;=\; -\nabla P + \nu\Delta\boldsymbol{v} + \boldsymbol{f},$$

Geostrophic part of the flow $\boldsymbol{U} = \boldsymbol{U}(t,x,y)$, fast changing and small-scale part $\boldsymbol{u}$, full velocity field $\boldsymbol{v} = \boldsymbol{U} + \boldsymbol{u}$:

$$\partial_t \boldsymbol{U} + (\boldsymbol{U}\cdot\nabla)\boldsymbol{U} + \langle(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\rangle_z \;=\; -\nabla P - \alpha\boldsymbol{U} + \nu\Delta\boldsymbol{U}$$

Effective bottom friction is maintained by secondary flow produced by Eckman suction in a narrow viscous sublayer

$$\delta_E = \sqrt{\mathrm{Ek}}\cdot H \qquad \alpha = \sqrt{\mathrm{Ek}}\cdot\Omega, \qquad \mathrm{Ek} = \frac{\nu H^2}{\Omega}.$$

Cyclone in a closed vessel

Thickness of boundary layer

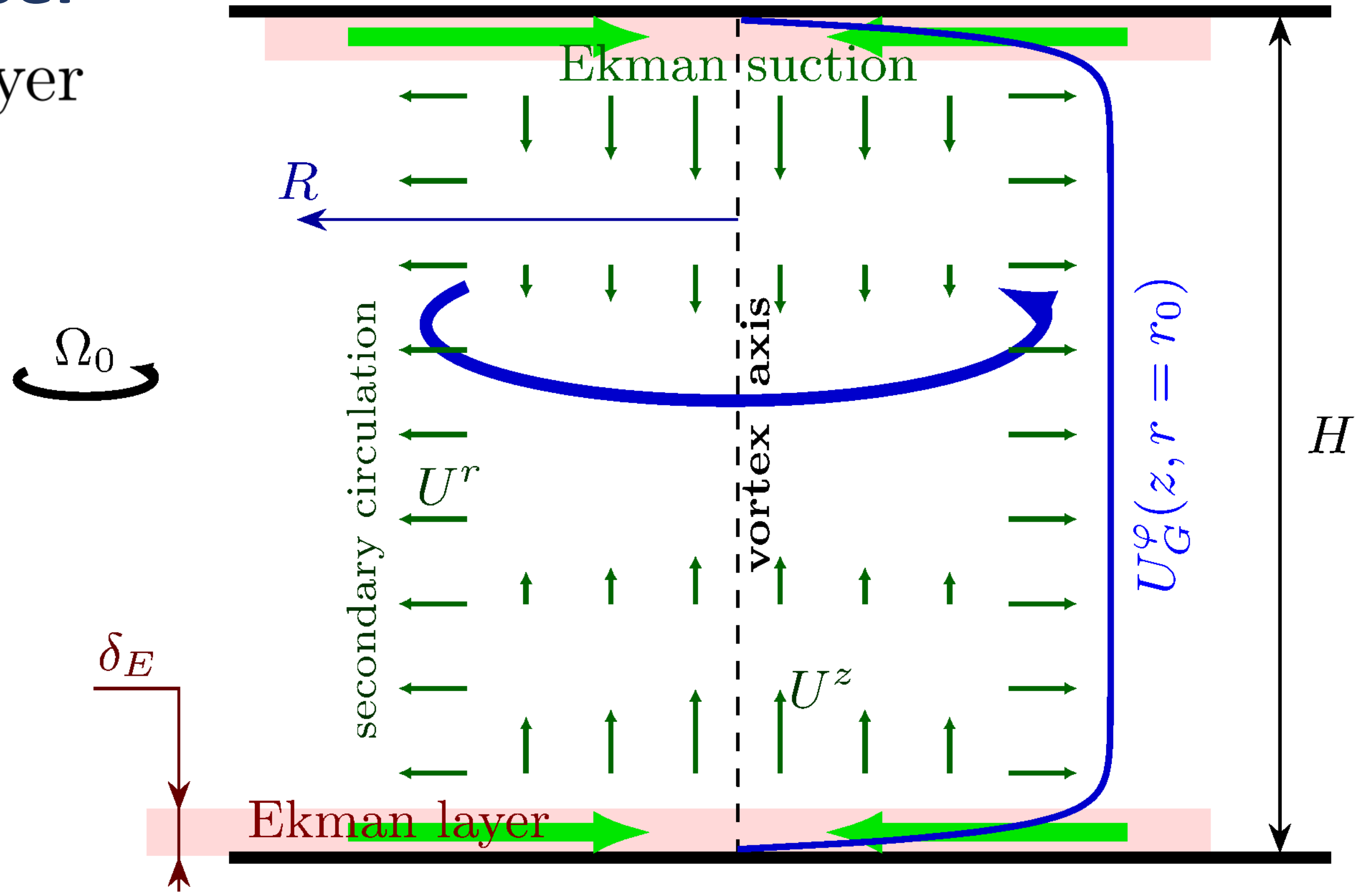
$$\delta_E = \sqrt{\mathrm{Ek}} \cdot H$$

Ekman number

$$\mathrm{Ek} = \frac{\nu H^2}{\Omega_0}.$$

Ekman suction

$$\delta U^\alpha = -\epsilon^{\alpha\beta} U^\beta$$



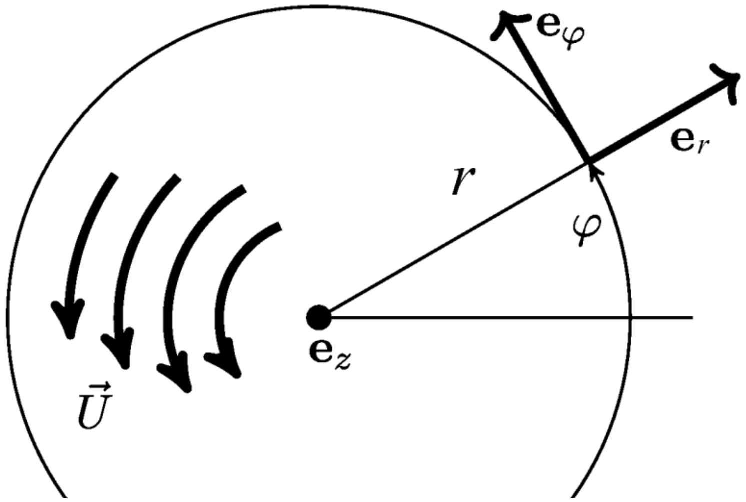
Secondary flow in volume:

$$\delta U^\alpha = \sqrt{\mathrm{Ek}} \cdot \epsilon^{\alpha\beta} U^\beta, \qquad \delta U^z = \left(\frac{1}{2} - \frac{z}{H}\right) \delta_E \epsilon^{\alpha\beta} \partial_\alpha U^\beta$$

Interaction between geostrophic flow and inertia waves

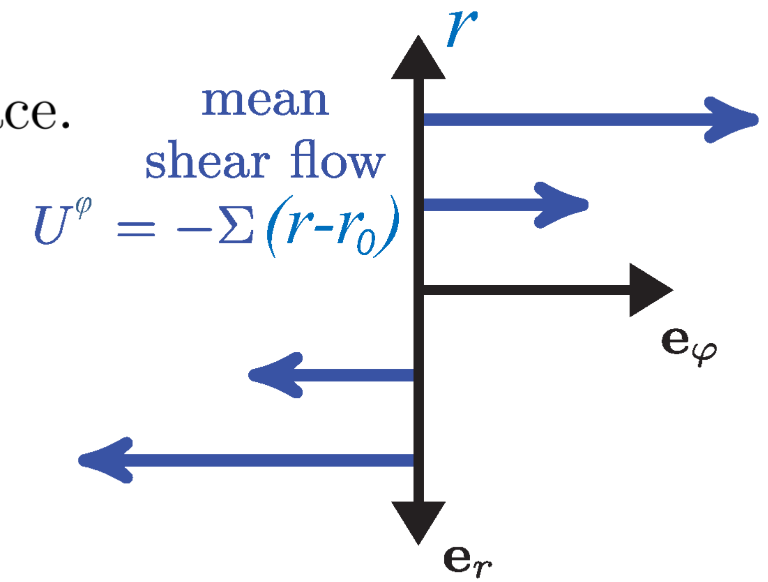
theory

Wave refraction in a geostrophic vortex flow



Quasi-lagrangian reference system, moving along circular orbit with radius r_0 with velocity $U(r_0)$ and rotating with angular velocity $U(r_0)/r_0$.

The vortex flow is locally shear flow $U^\varphi = \Sigma(r - r_0)$, where $\Sigma = r\partial_r(U/r)$. In our model, Σ is homogeneous in space.



Navier-Stokes equation linearazed in $\boldsymbol{u}$

$$\left(\partial_t - \Sigma k^\varphi \partial_{k^r}\right) u^i_{\mathbf{k}} + 2[\boldsymbol{\Omega} \times \boldsymbol{u}]^i + \Sigma u^r_{\mathbf{k}} \mathbf{e}_y =$$

$$= -\nu \Delta u^i_{\mathbf{k}} - i k^i p + f^i$$

Dynamics occurs along characteritics $k'^r(t)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{k}^{r\prime} = \Sigma\, k^\varphi .$$

Kolokolov et.al., PRF 5, 034604 (2020)
Gallet, JFM 783, 412 (2015)
Gelash et.al., JFM 831 p. 128 (2017)

Wave absorption by a geostrophic vortex flow

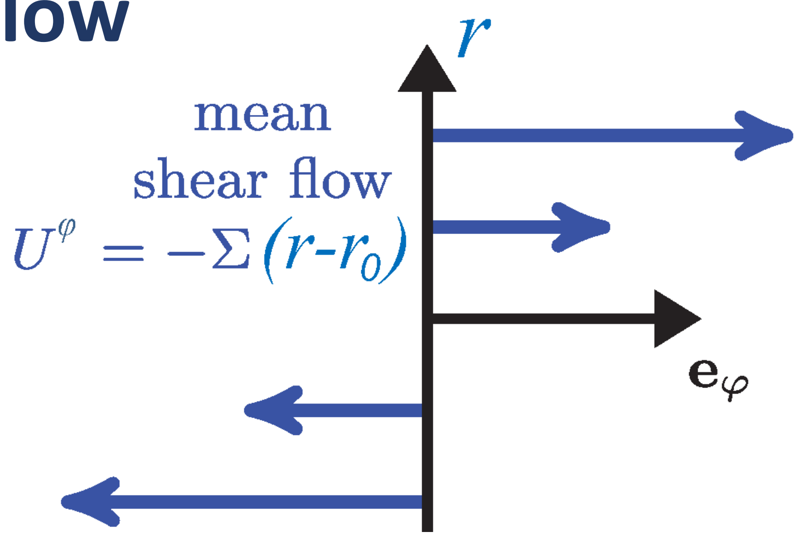
The moving along characteristics leads to change in frequency $k'^r(t)$

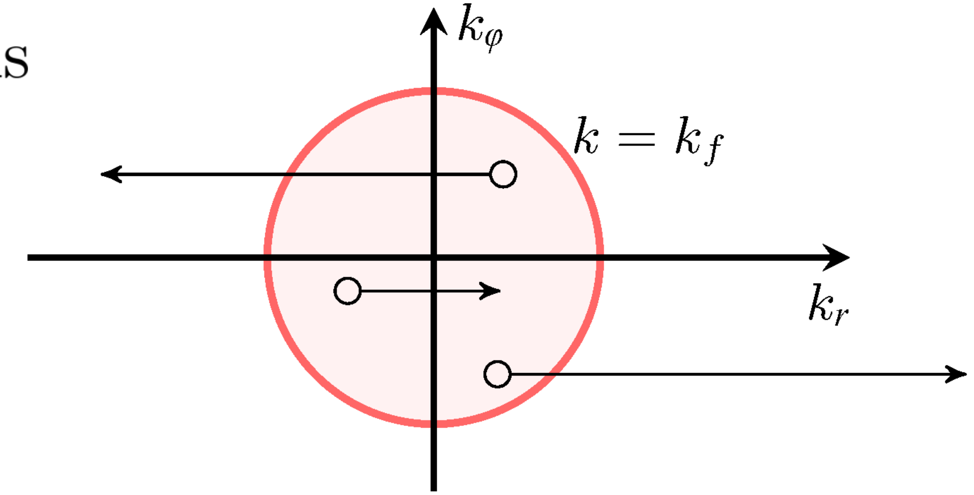
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{k}^{r\prime} = \Sigma\, k^{\varphi}, \qquad \omega_{\mathbf{k}'} = 2\Omega\frac{k^z}{k'}.$$

If Rossby number is small, $\omega_{\mathbf{k}'} \gg \Sigma$, then dynamics of two opposite polarizations is uncoupled, each polarization is an oscillator with adiabatically changing frequency:

$$\frac{E_{\mathbf{k}s}}{\omega_{\mathbf{k}'}} \propto k' E_{\mathbf{k}s} = \mathrm{inv}, \qquad E_{\mathbf{k}s} \propto 1/k'.$$

The energy $E_{\mathbf{k}s}$ stored in wave diminishes





Kolokolov et.al., PRF 5, 034604 (2020)
Gallet, JFM 783, 412 (2015)
Gelash et.al., JFM 831 p. 128 (2017)

Wave influence on the geostrophic flow

This is homogeneous in space problem, we assume the corresponding statistics of external force $\boldsymbol{f}$

$$\langle f^i_{\mathbf{k}}(t) f^j_{\mathbf{q}}(t')\rangle = \epsilon\,\delta(\mathbf{k}-\mathbf{q})\chi(k/k_f)$$

ϵ is the volume power of the force.

The wave produces mean force acting on the geostrophic flow.
The relevant component of Reynolds tensor

$$\langle u^r u^\varphi\rangle = \frac{\epsilon}{\Sigma}\int\limits_0^\infty \mathrm{d}\tau \int \frac{\mathrm{d}^3 q}{(2\pi)^3}\frac{q'^r q^\varphi}{q'^3}\chi(q)\,\exp\left(-\frac{\nu k_f^2}{\Sigma}\Gamma\right) = \frac{\epsilon}{\sigma}F, \qquad F < 1.$$

The integral is accumulated at times $t \sim 1/\Sigma$.

Equation for the mean flow

Mean radial profile of azimuth velocity U, the shear rate in the differential rotation $\Sigma = r\partial_r(U/r)$:

$$\partial_t U + \left(\partial_r + \frac{2}{r}\right)\left(\langle u^r u^\varphi\rangle - \nu\Sigma\right) \;=\; -\alpha U$$

The tangent Reynolds stress $\langle u^r u^\varphi\rangle = F\dfrac{\epsilon}{\Sigma}$, "efficiency" $F < 1$.

Energy balance:

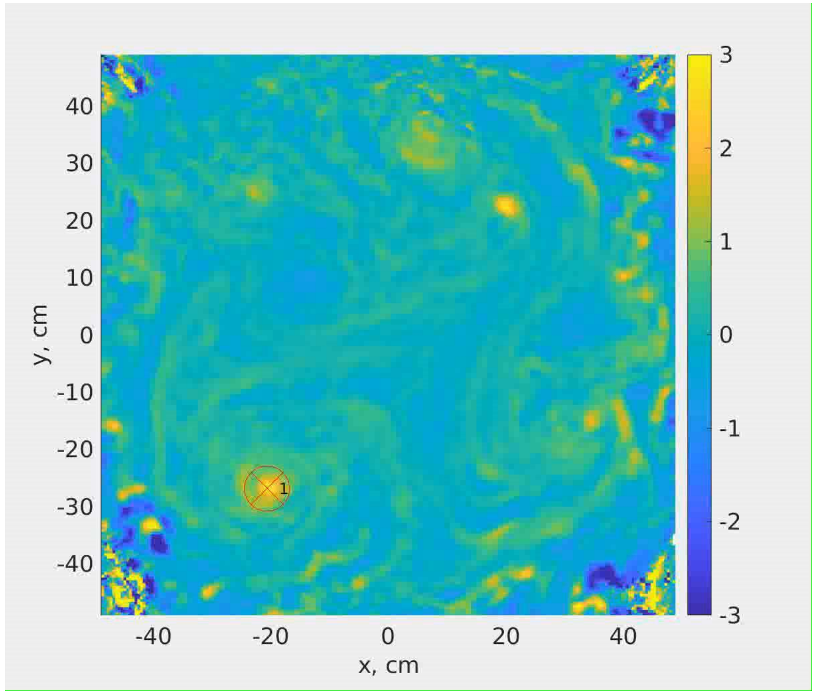
$$\frac{\partial_r(rJ^r)}{r} = F\epsilon - \nu\Sigma^2 - \alpha U^2, \qquad \text{flux } J^r = U\left(\langle u^r u^\varphi\rangle - \nu\Sigma\right).$$

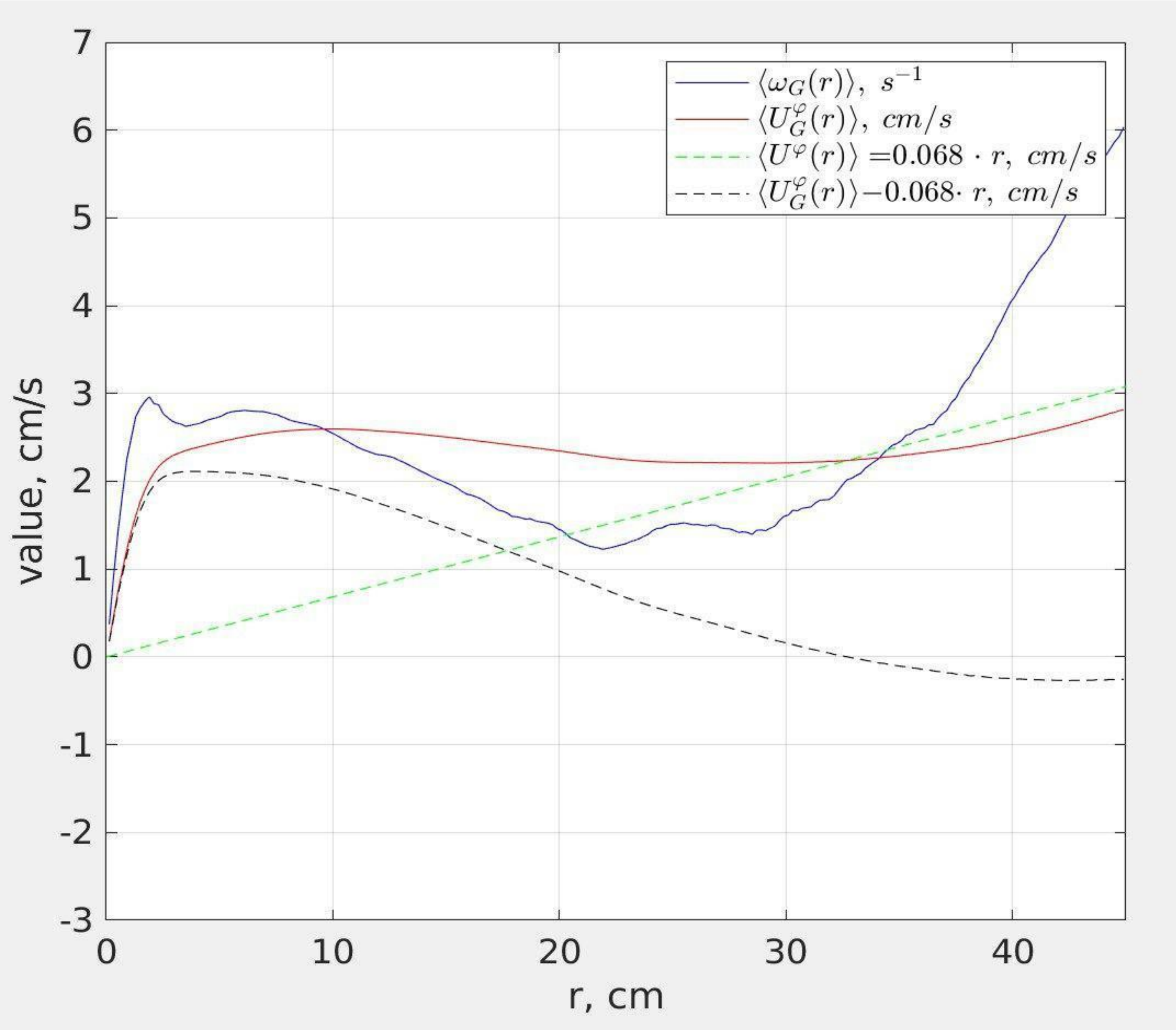
Characteristic scale $R_\alpha = \sqrt{\nu/\alpha}$. If $F \approx 1$, and distances $r \gg R_\alpha$ (bottom friction dominates), then the profile

$$U = \sqrt{3\epsilon/\alpha}$$

Moderate rotation speed, 0.12 Hz

The velocity profile U(r) has a plato (red curve)



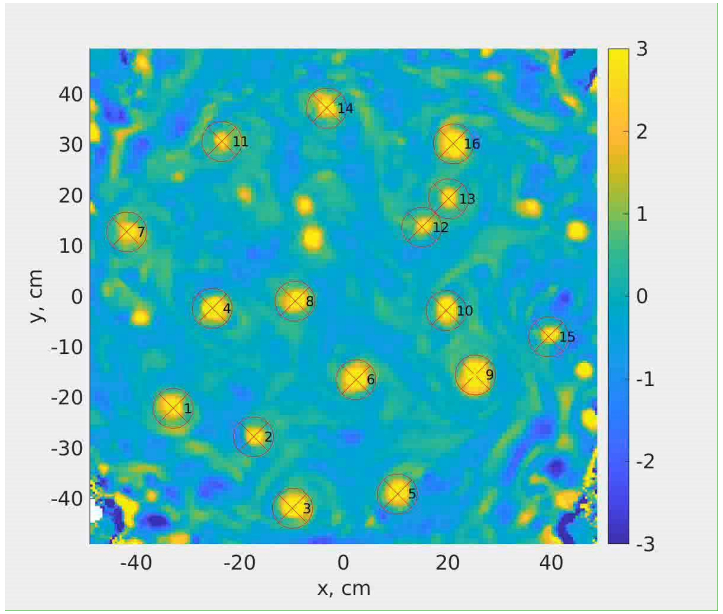


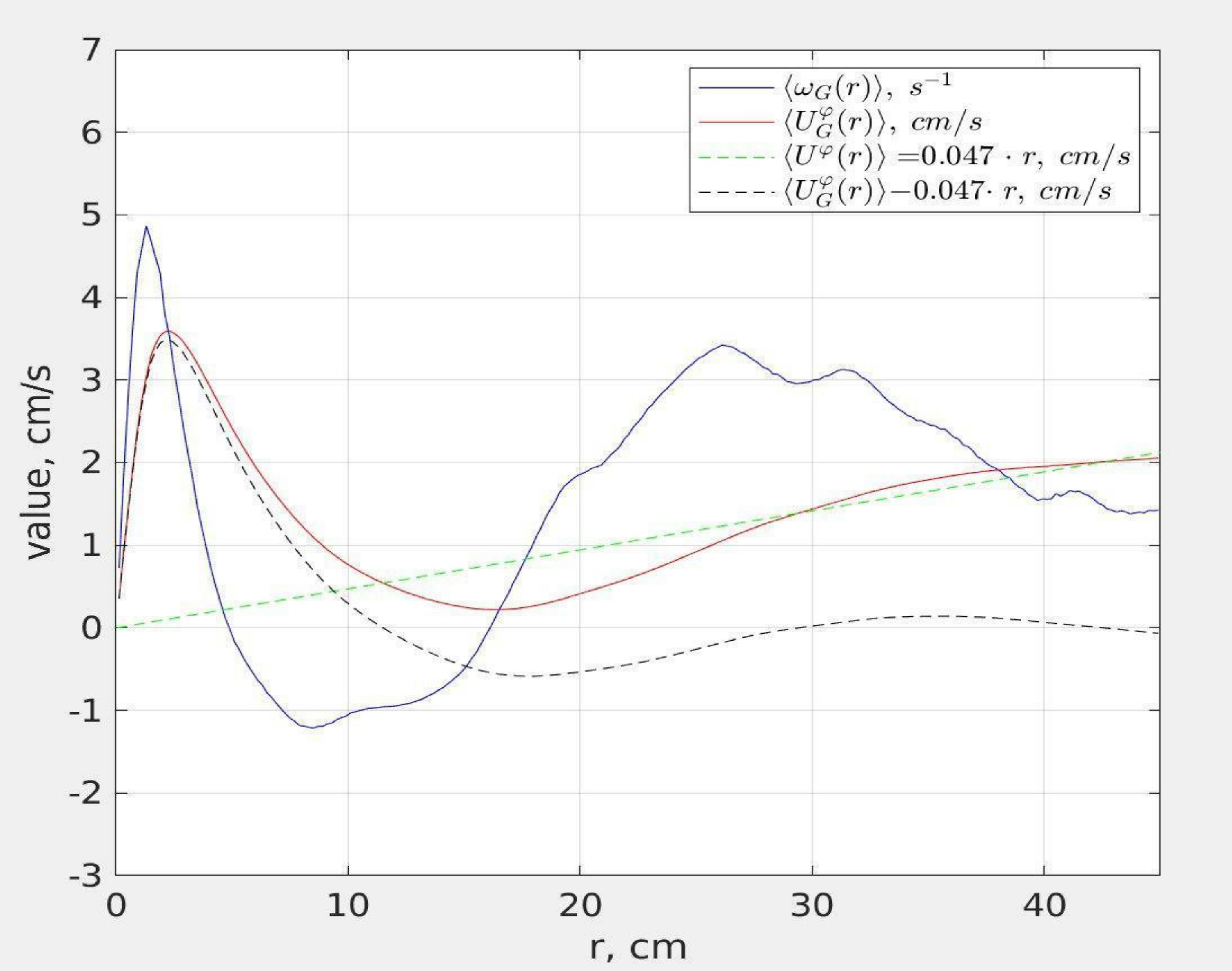
Fast 0.7 Hz vs. moderate rotation 0.12 Hz

theory

Fast rotation speed, 0.7 Hz

Vortices are isolated, that is full circulation associated with a vortex is small





Scales

Wave:

- Wavelength $\lambda = 2\pi/k$.
- Group velocity $v_g \sim \lambda\Omega$.
- Alternation of wavevector in inhomogeneous geostrophic flow:

$$\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{k} \sim \frac{1}{\Omega}\frac{\mathrm{d}}{\mathrm{d}r}U.$$

Scales:

- Wavelength $\lambda = 2\pi/k$.
- Scale of geostrophic flow $L_U \gg \lambda$.
- Scale L_w (if exists) of wave absorption, due to wave travels in inhomogeneous geostrophic flow $\boldsymbol{U}$. The velocity changes on $\Delta U \sim \Omega/k \sim v_g$ at scale L_w.

Effective wave absorption: $\mathrm{Ro}_M \equiv \frac{U_{rms}}{2\Omega\, L_f} > \mathrm{Ro}^*_M$, L_f - scale of mixers

Possible problem formulations:

- Homogeneity in space, heterogeneity in time
- Homogeneity in time, inhomogeneity in space

Wave generation by a localized source

Amplitude of a monochromatic wave which travels into a vortex

$$t(r) \propto \frac{k}{\sqrt{|k_y|}} t_0, \qquad \mathrm{d}\left(\frac{1}{k}\right) = -\frac{s k^{\varphi}}{k^z \cdot 2\Omega} \mathrm{d}U, \qquad \Sigma = \partial_r U(r),$$

where $\mathbf{k} = \{k^r(r), k^{\varphi}, k^z\}$ and $s = \pm 1$ is the wave polarization,
и $\{r, \varphi, z\}$ — ~~cylindrical~~ Cartesian coordinates.

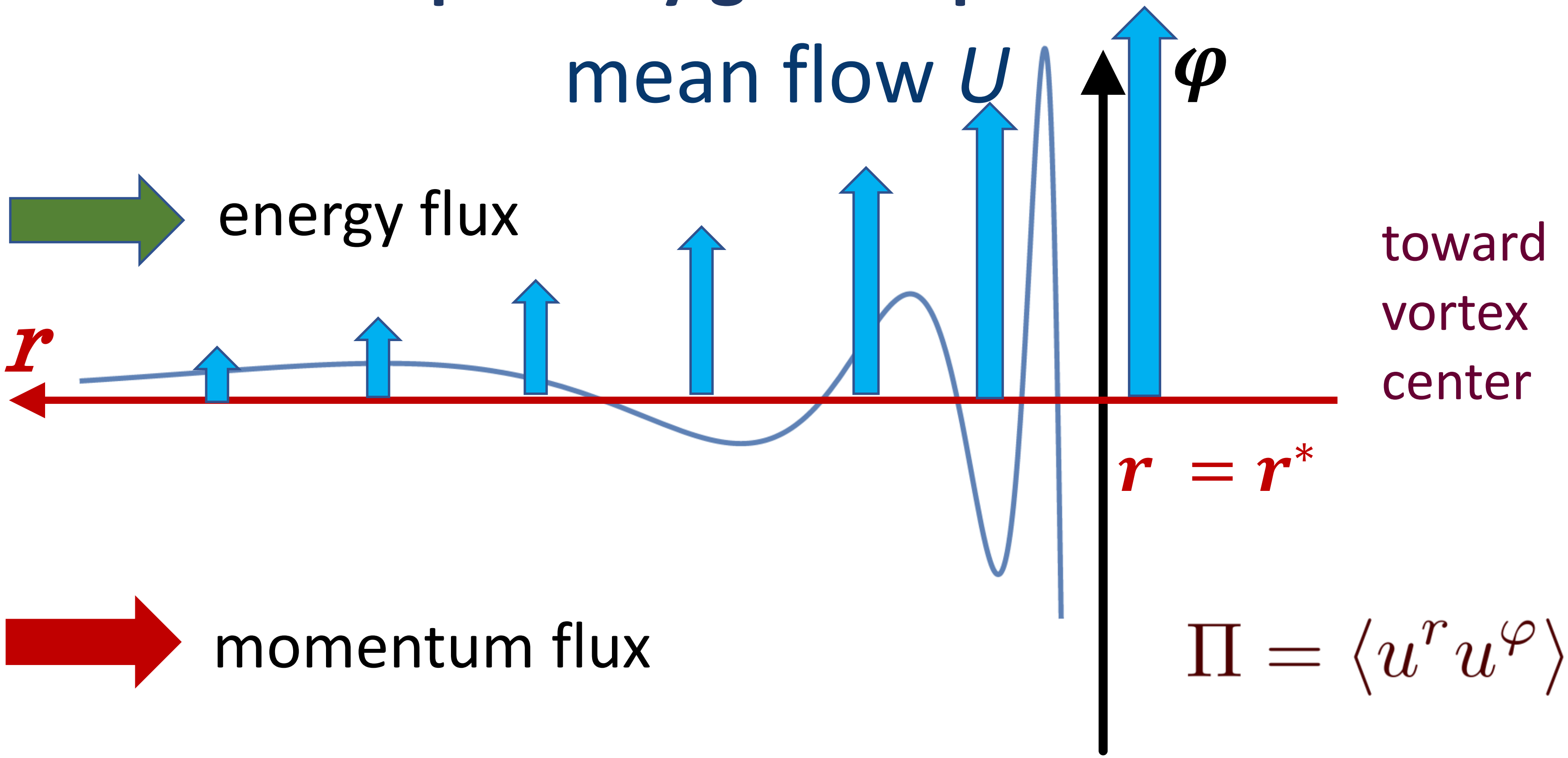
Singular point r_*: when approaching, wave vector $k \to \infty$, further wave traveling is impossible.

Before the point at $r < r_*$, the fluxes of energy J and φ-momentum Π remain constant,

$$J = \langle u^{\varphi} P + U u^r u^{\varphi} \rangle = \mathrm{const}, \qquad \Pi = \langle u^r u^{\varphi} \rangle = \mathrm{const.},$$

Transfer of momentum and energey from wave to geostrophic flow accurs in plane $r = r_*$.

Wave absorption by geostrophic flow

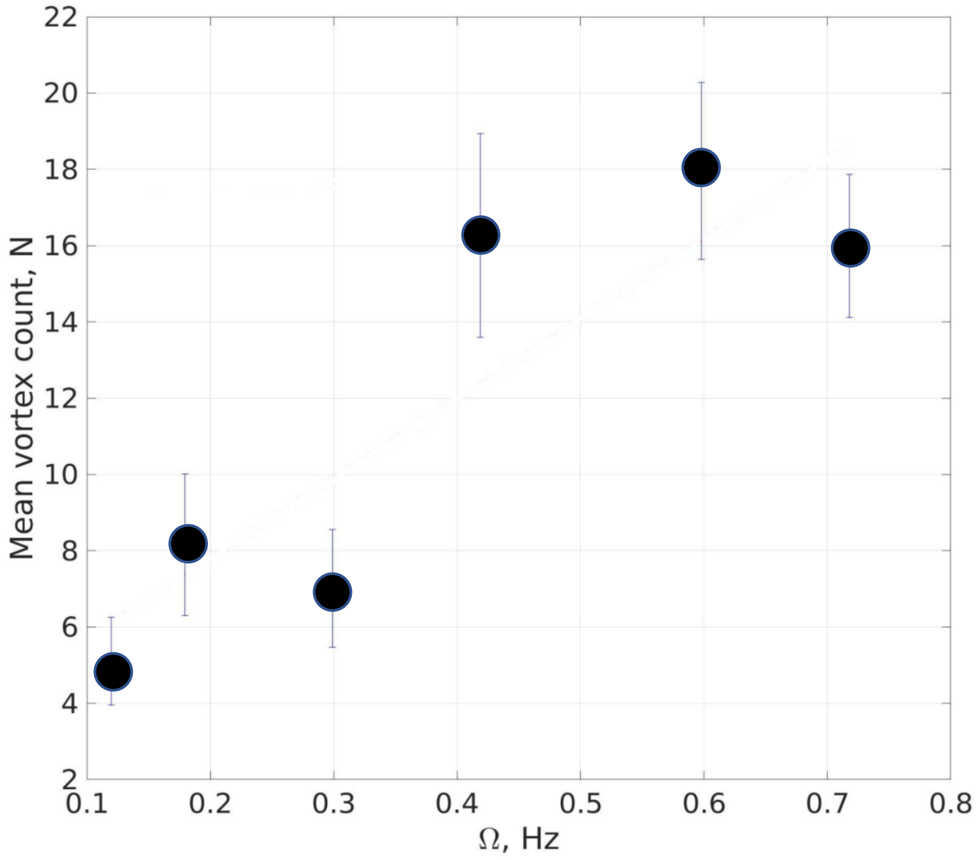


$$\Pi = \langle u^r u^\varphi \rangle = \Pi_0\, \theta(r - r^*),$$

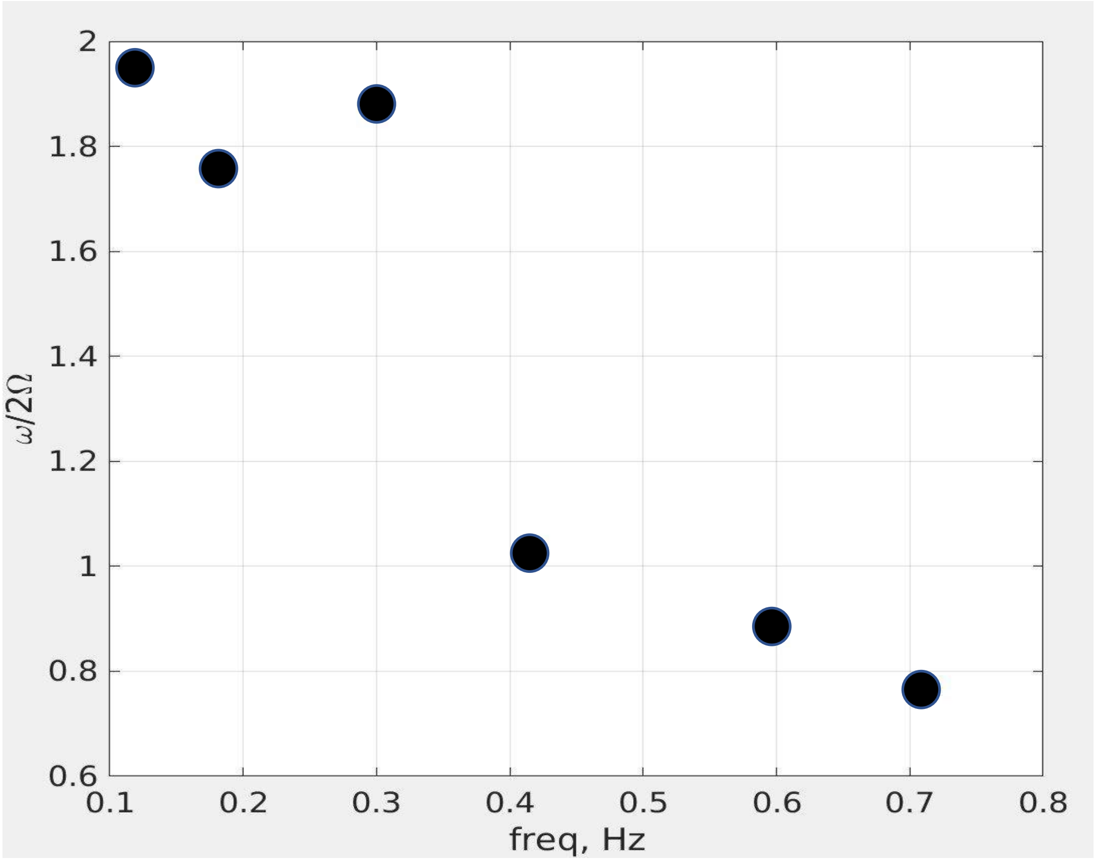
Navier-Stokes equation averaged over wave oscillations

$$\partial_t U = -\partial_r \langle u^r u^\varphi \rangle = \Pi_0\, \delta(r - r^*), \qquad \Pi_0 > 0$$

Two regimes of vortex motion



Number of vortices



Rossby micronumber

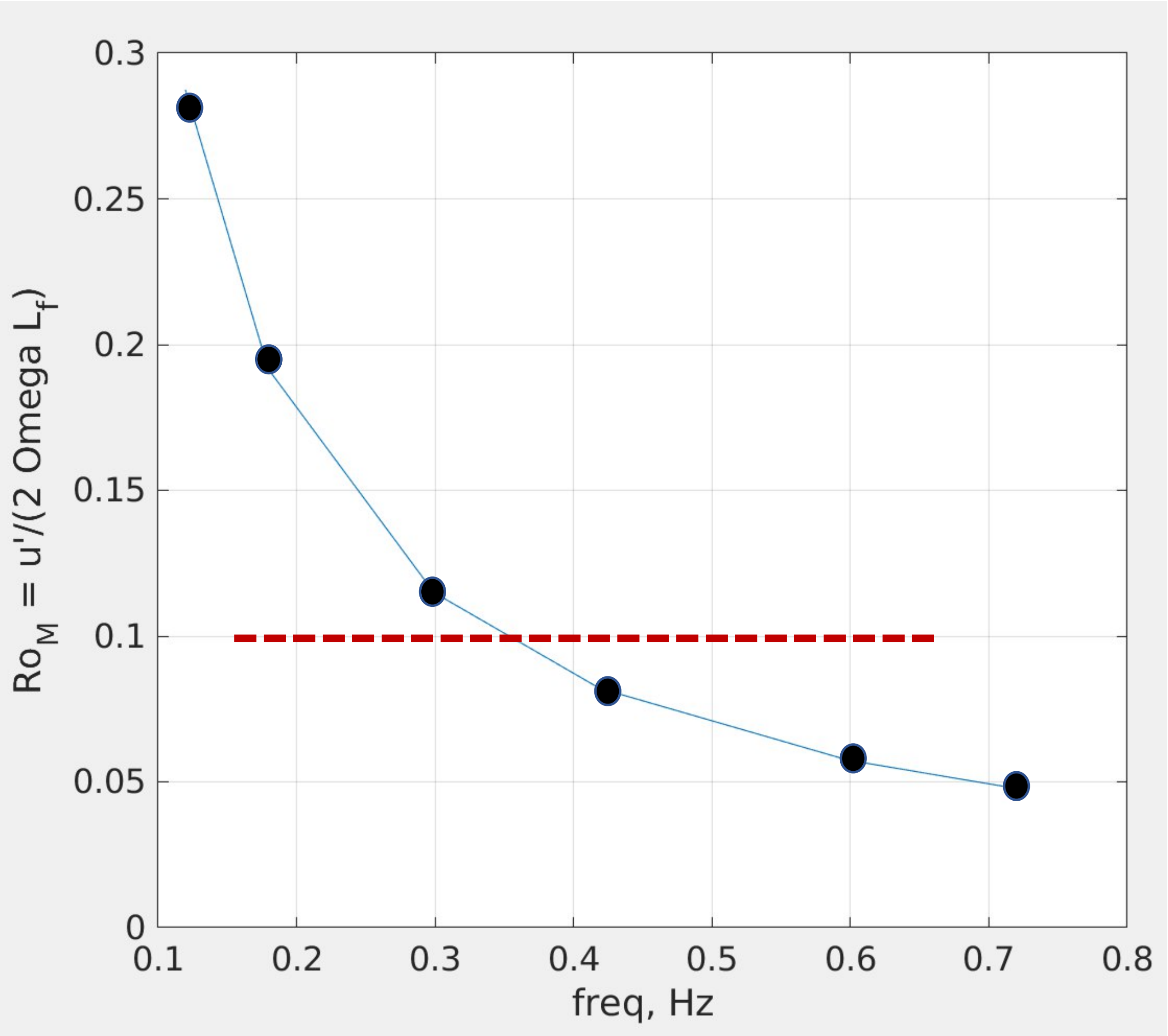
$$Ro_m = \frac{\omega_{max}}{2\Omega}$$

Transition governed by Rossby macro number

$$Ro_M = \frac{U_{rmc}}{2\Omega\, L_f}$$

Transition at $Ro_M^* = 0.1$

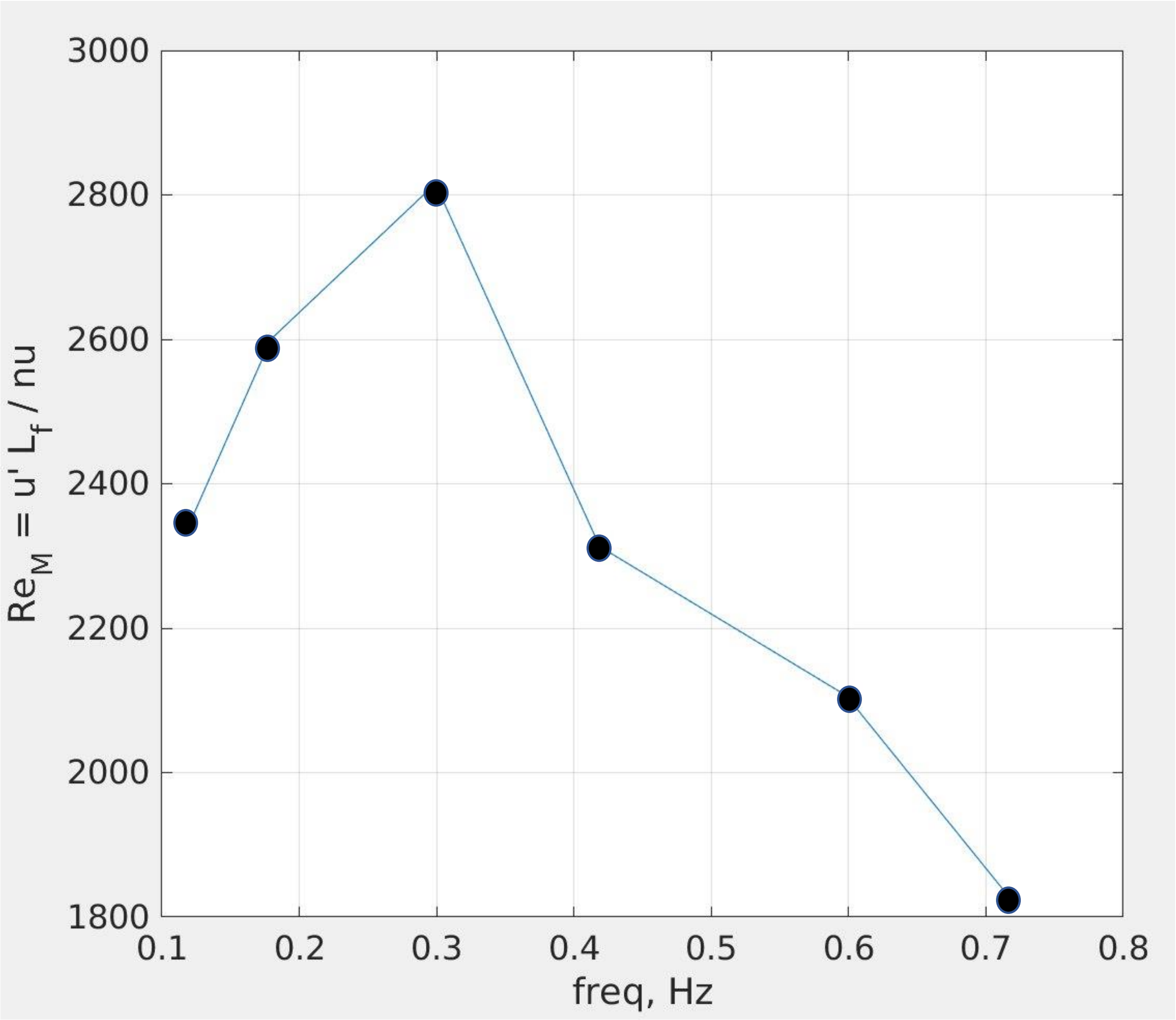
Above transition, at $Ro_M > Ro_M^*$, waves are absorbed by vortices, Below it, at at $Ro_M < Ro_M^*$, the effectiveness of the absorption is suppressed.



Reynolds macro number

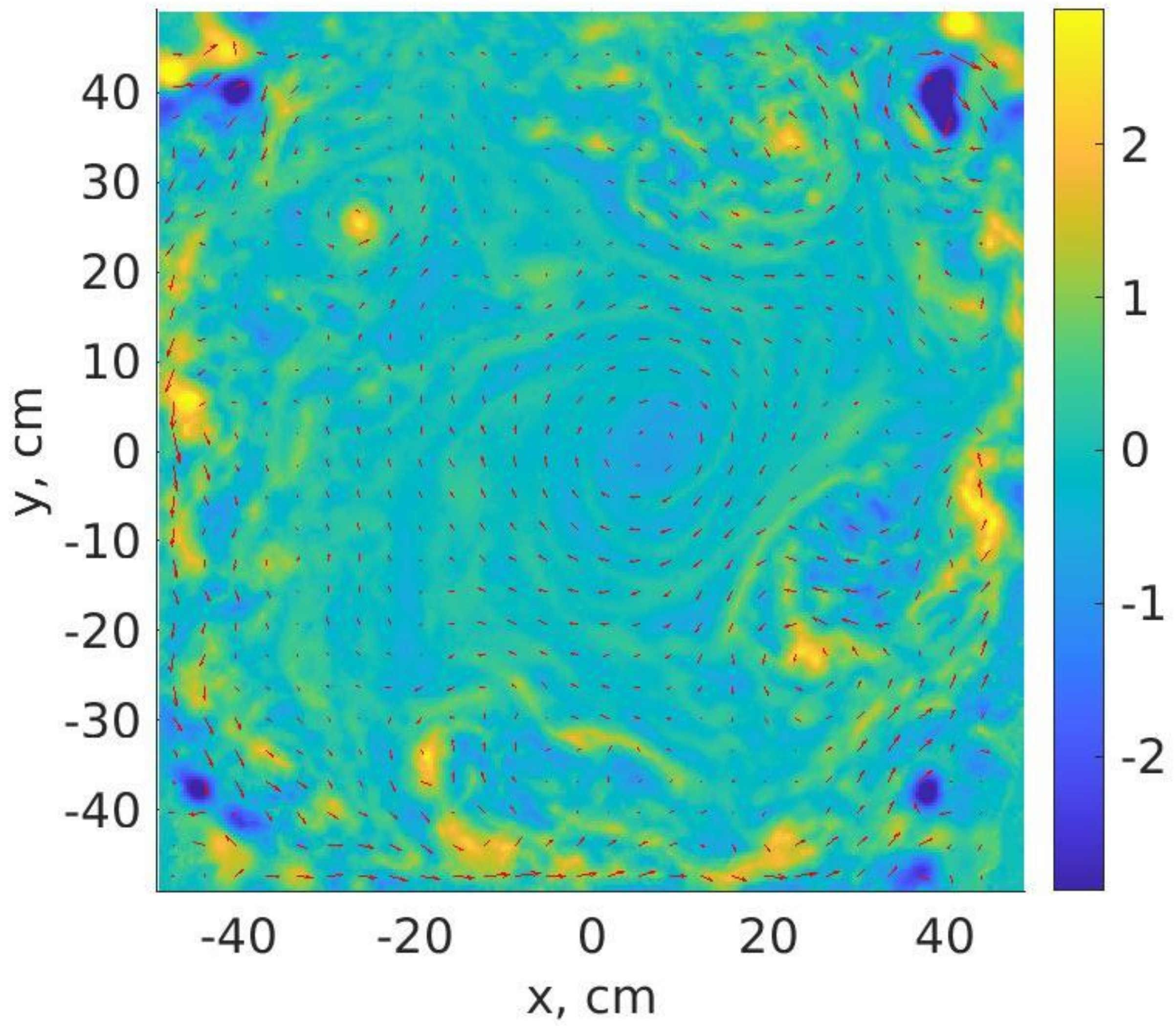
$$Re_M = \frac{U_{rmc}\, L_f}{\nu}$$

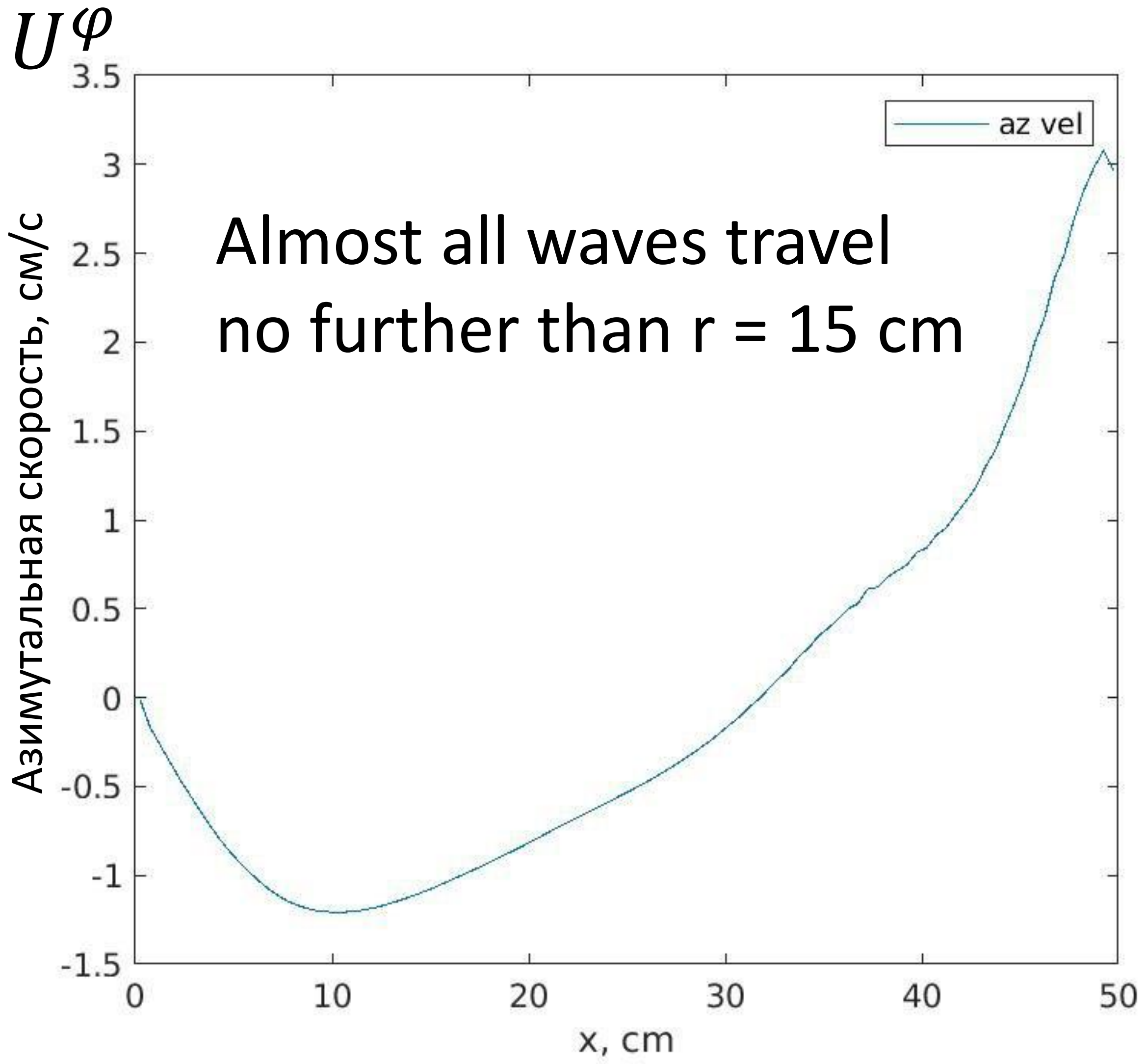
ν is kinematic viscosity



Extremely slow rotation 0.06 Hz

One large anti-cyclone





Almost all waves travel no further than r = 15 cm

Results:

1. Different regimes of geostrophic turbulence is observed in experiment
2. Developed theory provides explanation for the source of the difference between the regimes
3. Current theory does not explain sharp transition between the regimes

Publications

[1] V.M. Parfenyev, S.S. Vergeles, Influence of Ekman friction on the velocity profile of a coherent vortex in a three-dimensional rotating turbulent flow, Physics of Fluids 33, 115128 (2021)

[2] V.M. Parfenyev, I.A. Vointsev, A.O. Skoba, S.S. Vergeles, Velocity profiles of cyclones and anticyclones in a rotating turbulent flow, Physics of Fluids 33, 065117 (2021)

[3] I.V. Kolokolov, L.L. Ogorodnikov, S.S. Vergeles, Structure of coherent columnar vortices in three-dimensional rotating turbulent flow, Phys. Rev. Fluids 5, 034604 (2020)

[4] L.L. Ogorodnikov, S.S. Vergeles, Velocity structure function in a geostrophic coherent vortex under strong rotation, arXiv:2112.05976

[5] N.A. Ivchenko, S.S. Vergeles, Waves in a coherent two-dimensional flow, Physics of Fluids 33, 105102 (2021)

Thank you for your attention!