

Coherent vortices in two-dimensional pumped turbulence

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Turbulence is a chaotic eddy state excited in fluids due to external forcing at large Reynolds numbers Re . From the theoretical point of view turbulence is a system with strong interaction. No any perturbation approach can be used for analyzing turbulence. One can rely on relations based on conservation laws.

The general scheme of *3d* turbulence is based on energy transformations. Energy is pumped to the fluid at some scale called integral one. Due to the non-linear interaction eddies of smaller and smaller scales are produced. The process is terminated thanks to viscosity at some scale called viscous one. One says about the energy cascade from the integral scale to viscous one.

In the inertial interval between the integral state and the viscous one the flow statistics is dominated by non-linearity. The statistics is homogeneous and isotropic, and is characterized by scaling laws: all correlation functions of velocity are power functions of the separations between the points. Multifractality. The exponents are universal.

In thin fluid films turbulence is effectively two-dimensional on scales larger than the thickness of the film. The most interesting thin fluid film – atmosphere. We consider the simple model – two-dimensional Navier-Stokes equation – enabling one to formulate some qualitative conclusions concerning such complicated objects as atmosphere.

Surprisingly, two-dimensional turbulence appears to be more complicated than three-dimensional one. It is related to some additional quadratic integral of motion conserved by the dissipationless hydrodynamic equations – enstrophy. An interplay of the energy and the enstrophy flows leads to the so-called two-cascade picture of *2d* turbulence.

Two-dimensional hydrodynamics is described by the equation for vorticity $\varpi = \text{curl } \boldsymbol{v}$

$$\partial_t \varpi + \boldsymbol{v} \nabla \varpi = \text{curl } \boldsymbol{f} + \nu \nabla^2 \varpi - \alpha \varpi,$$

where $\boldsymbol{v}$ is velocity, $\boldsymbol{f}$ is pumping force per unit mass, ν is viscosity and α is bottom friction coefficient. The flow is assumed to be incompressible: $\text{div } \boldsymbol{v} = 0$.

The turbulent state is realized if the forcing is strong enough to overcome dissipation. First, Reynolds number should be large. It is defined as $\text{Re} = v_f / (\nu k_f)$ where k_f^{-1} is the characteristic scale of forcing and v_f is the velocity at the scale. In addition, the inequality $\alpha \ll k_f v_f$ should be satisfied. Then dissipation is irrelevant in some interval of scales.

If we discard dissipation and forcing, then there are two quadratic integrals of motion, energy and enstrophy:

$$\mathcal{E} = \int dx \, dy \, v^2, \quad \mathcal{S} = \int dx \, dy \, \varpi^2.$$

The external force produces energy $\dot{\mathcal{E}}$ and enstrophy $\dot{\mathcal{S}}$, related as $\dot{\mathcal{S}} \sim k_f^2 \dot{\mathcal{E}}$. The non-linear interaction transfers both to the dissipation scales.

At the conditions the scales are far from k_f^{-1} . Therefore in the steady state the energy and the enstrophy cannot dissipate at the same scale. Steady turbulence – two cascades: enstrophy flows to small scales whereas energy flows to large scales, being dissipated by viscosity and friction, respectively (Kraichnan 1967, Leith 1968, Batchelor 1969).

The relation for the third-order correlation function for $3d$ was established by Kolmogorov (1941). Adopting it for $2d$, one obtains

$$\begin{aligned}\langle |v_1 - v_2|^3 \rangle &\propto r, & r \gg k_f^{-1}, \\ \langle (v_1 - v_2)\varpi_1\varpi_2 \rangle &\propto r, & r \ll k_f^{-1},\end{aligned}$$

where angular brackets designate temporal mean. The relations reflect constancy of the energy and enstrophy fluxes, respectively.

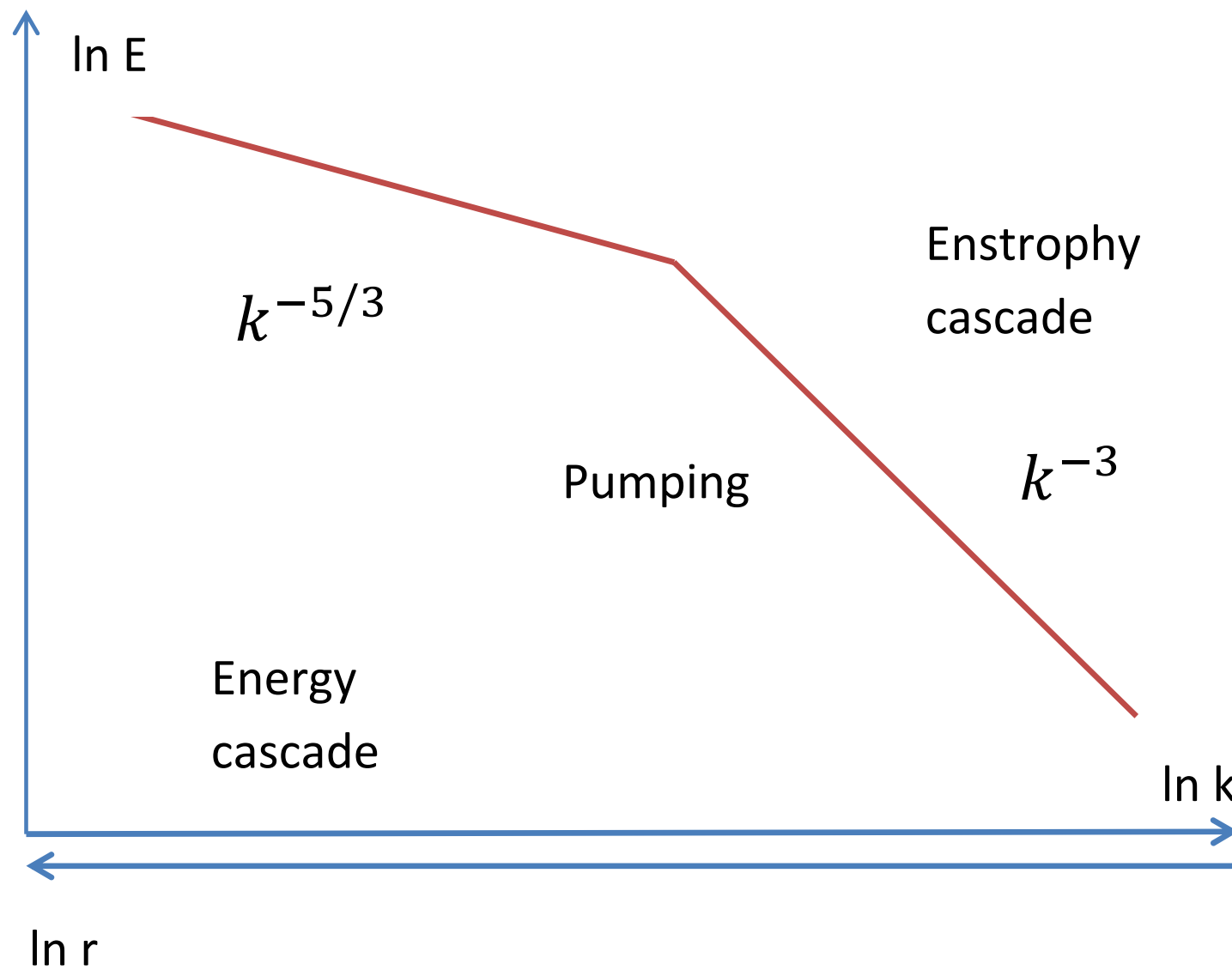
If we assume that the correlation functions are functions solely on the current scale then we arrive at the normal scaling $v_1 - v_2 \propto r$ in the direct cascade and $v_1 - v_2 \propto r^{1/3}$ in the inverse cascade. As opposed to $3d$, in $2d$ the normal scaling is valid for the inverse cascade (Paret and Tabeling 1998, Boffetta, Celani and Vergassola 2000).

The spectrum $E(k)$ is defined as

$$\langle v_1 v_2 \rangle = \int \frac{dk}{2\pi} e^{ikr} E(k).$$

The normal scaling implies $E(k) \propto k^{-3}$ for the direct (enstrophy) cascade $E(k) \propto k^{-5/3}$ for the inverse (energy) cascade.

There are logarithmic corrections to the spectrum for the direct cascade.



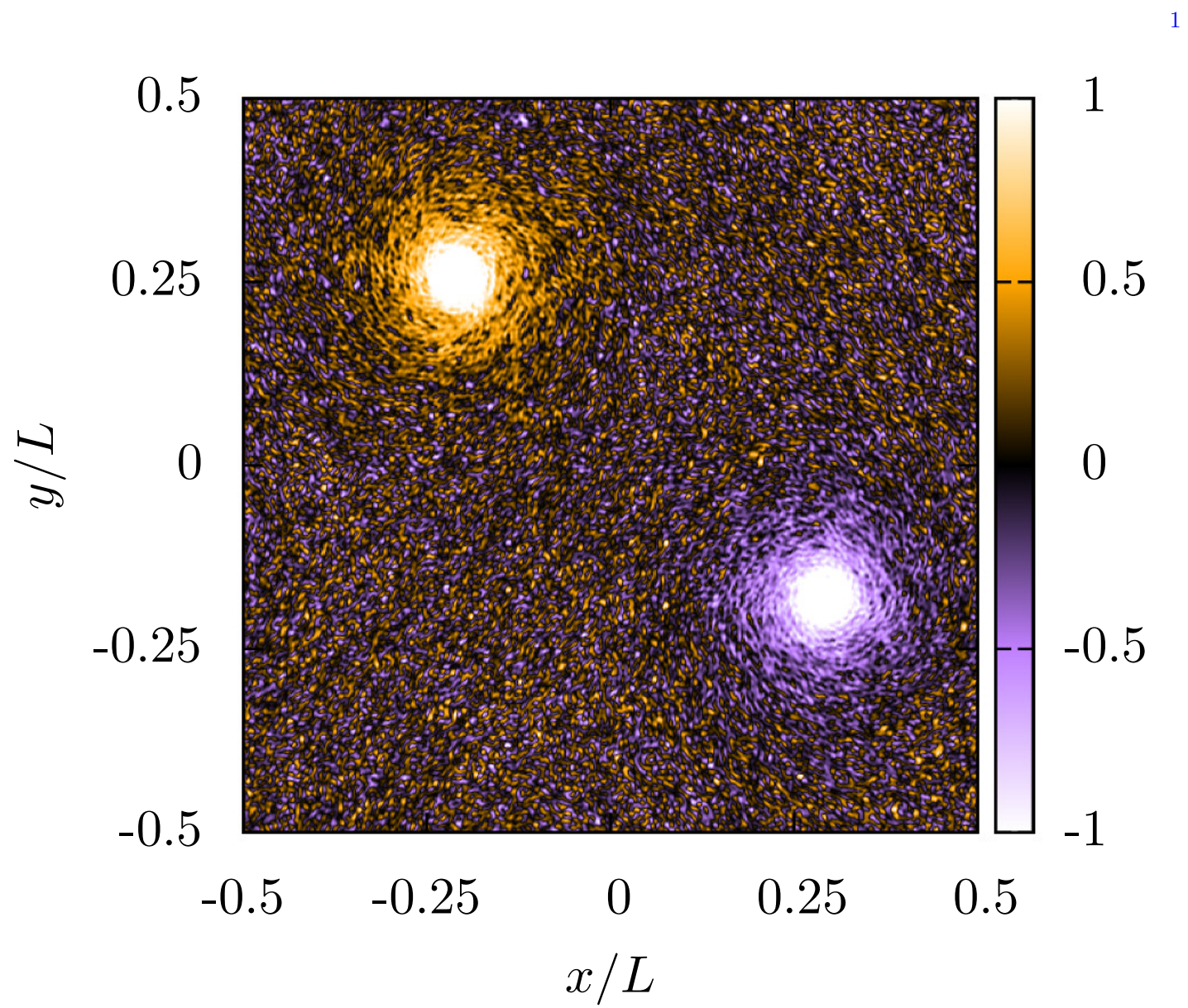
In an unbounded system the inverse cascade is terminated by the friction at the scale $L_\alpha \sim \epsilon^{1/2} \alpha^{-3/2}$ where ϵ is the energy production rate per unit mass. In the interval of scales $L_\alpha > r > k_f^{-1}$ Kolmogorov estimate for the velocity difference $v_1 - v_2 \sim (\epsilon r_{12})^{1/3}$ is correct. For scales larger than L_α the flow fluctuations are negligible.

If the size box $L < L_\alpha$ then the energy carried by the inverse cascade accumulates at L and is dissipated there by bottom friction. It leads to the estimate $v \sim (\epsilon/\alpha)^{1/2}$ for the velocity of large-scale flow fluctuations, with scale $\sim L$. The value is much larger than Kolmogorov estimate. One can say that the inverse cascade is deformed.

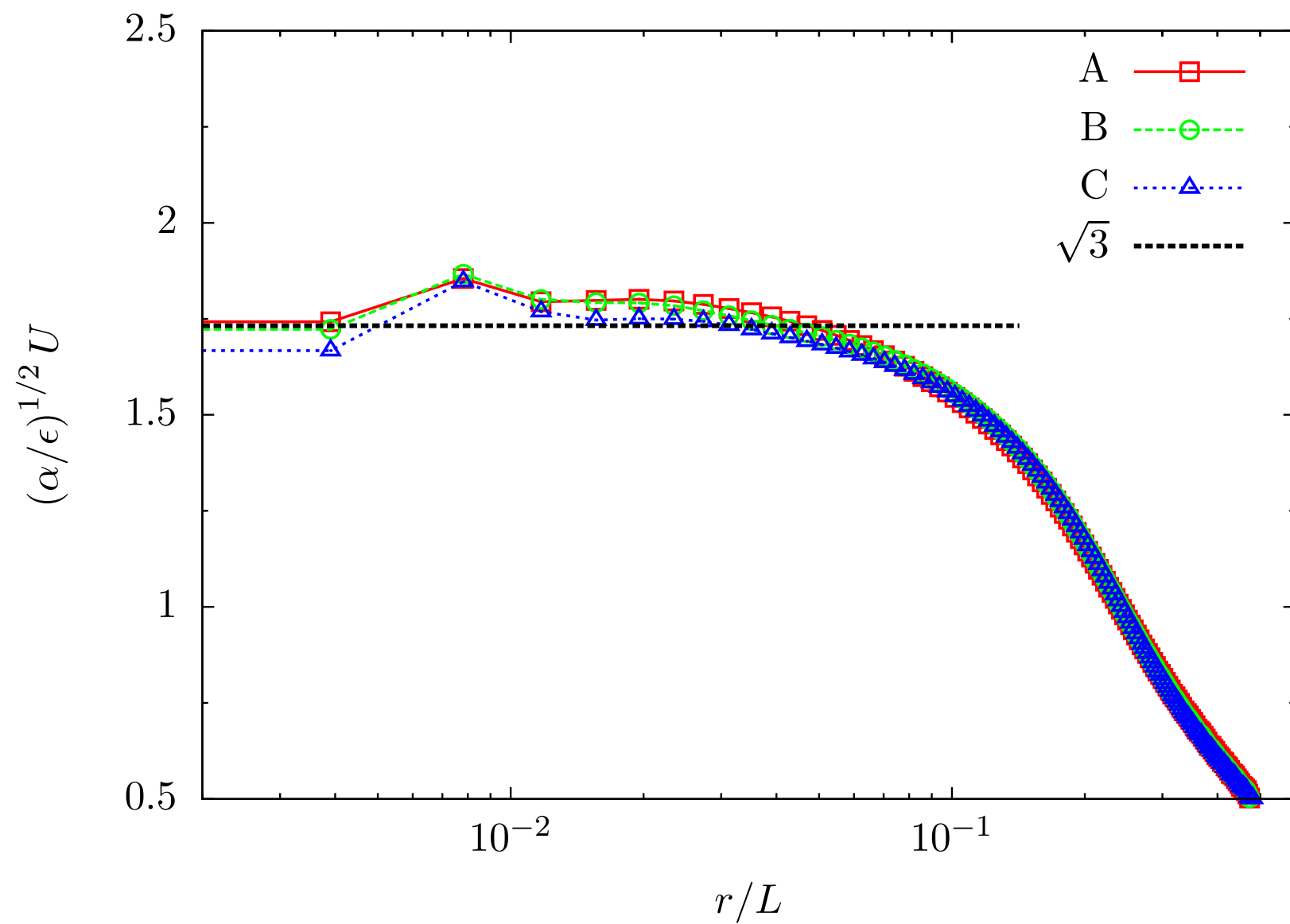
It is hard to conduct direct numerical simulations (DNS) of $3d$ turbulence due to a huge number of the variables. However, DNS of $2d$ turbulence can be conducted with reasonable efforts. The simulations (see, say, Molenaar, Clercx and van Heijst 2004) confirm production of intensive large-scale eddies in a finite box.

A striking feature of $2d$ turbulence in a finite box is possibility to form coherent vortices. They were observed in numerical simulations (Chertkov, Connaughton, Kolokolov and Lebedev 2007) and in laboratory experiment (Shats, Xia, Punzmann and Falkovich 2007). The coherent vortices are characterized by long life time.

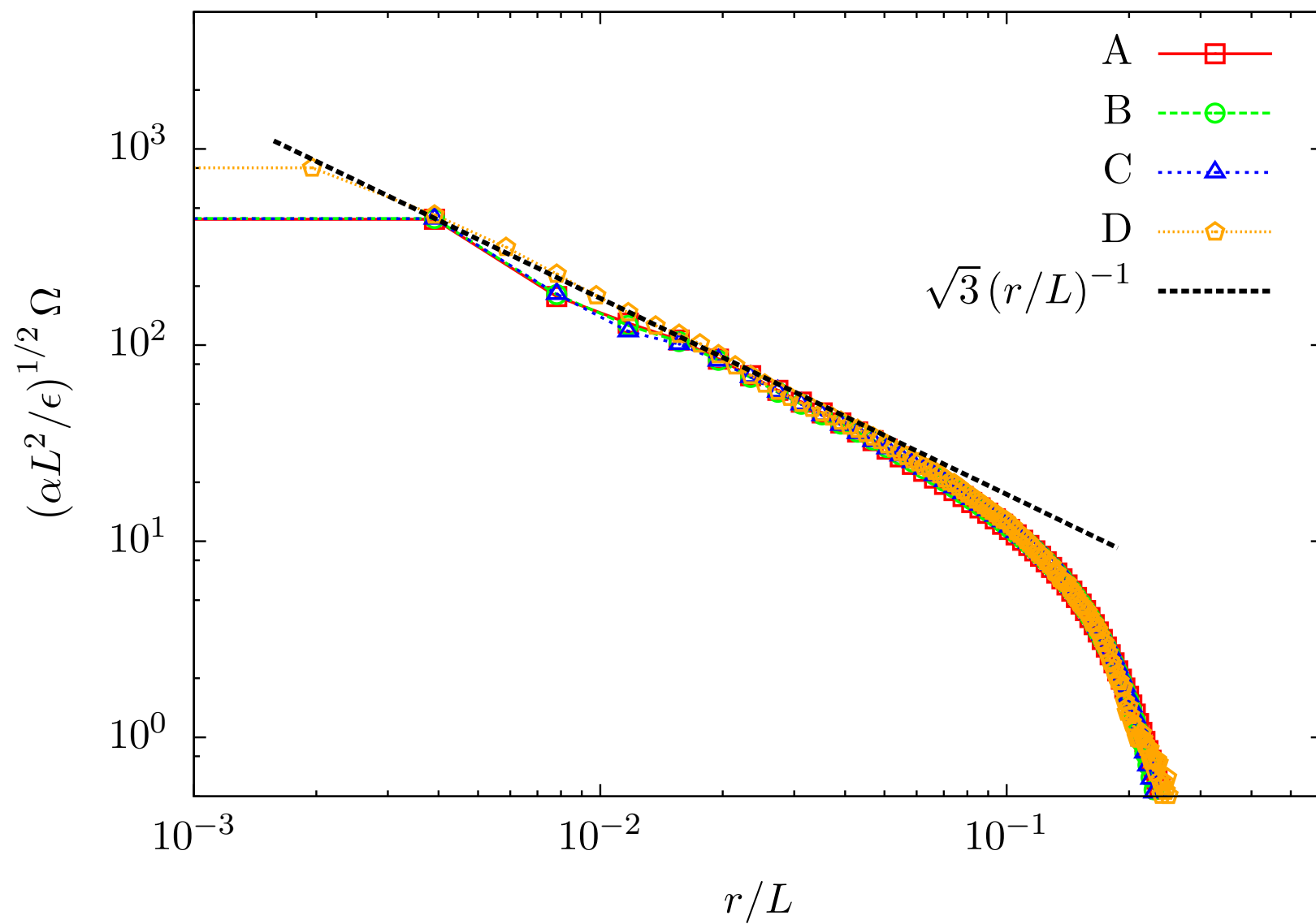
The coherent vortices can be characterized by a well defined mean profile with fluctuations suppressed on the top of the mean profile. We present some results of the numerical simulations of $2d$ pumped turbulence (Laurie, Boffetta, Falkovich, Kolokolov, Lebedev 2014). We give a typical snapshot of vorticity in the simulations.



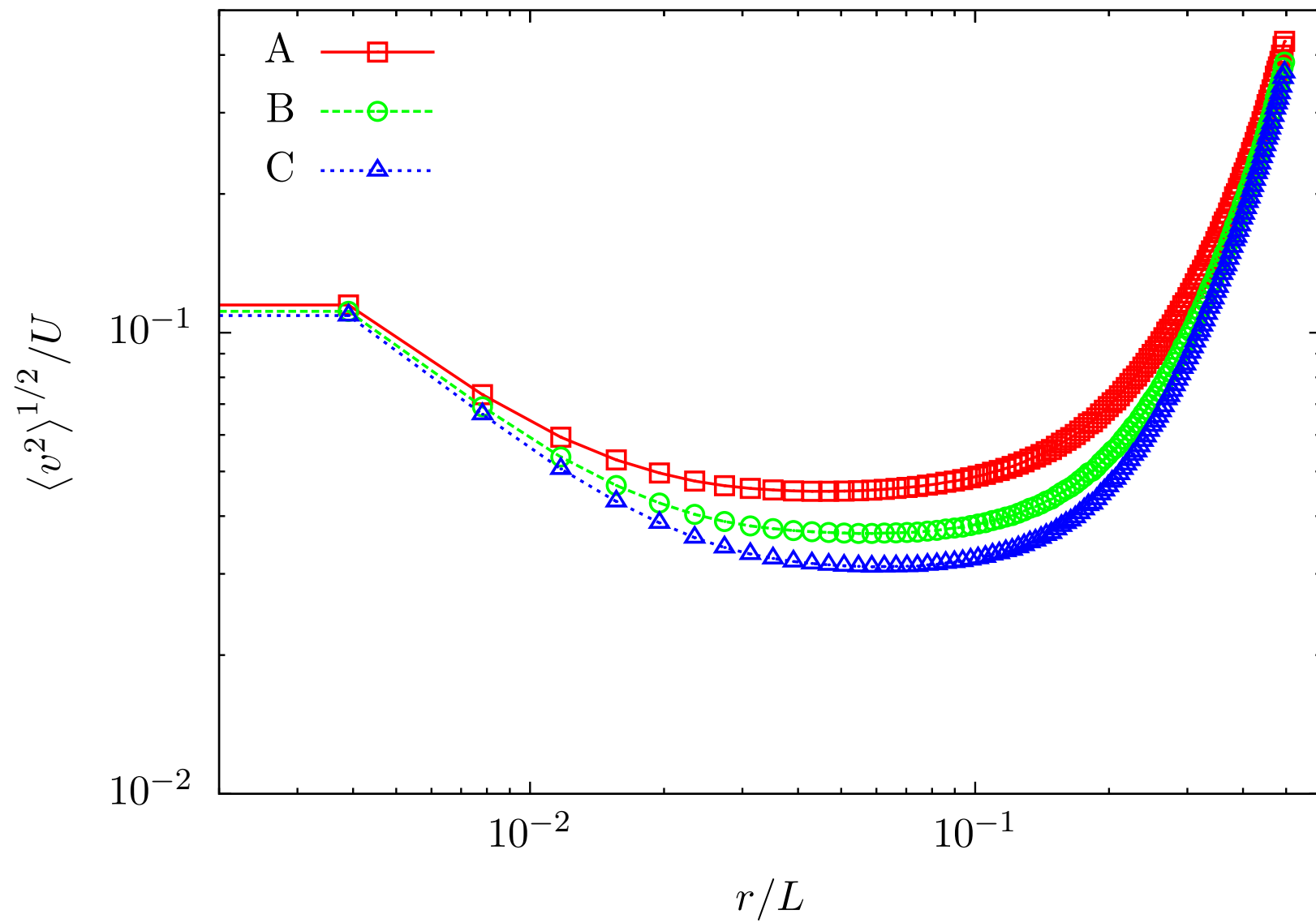
The simulations are conducted for the periodical setup. The flow is excited by random statistically steady pumping short correlated in time. We see the vortex dipole with the hyperbolic flow between the vortices. The dipole structure is explained by zero integral vorticity. The picture reveals fluctuations of different scales.



One can extract the mean velocity profile of the big vortex in the reference frame attached to the vortex center (point with maximal vorticity). It appears to be isotropic and can be characterized by the mean polar velocity U that is a function of the separation from the vortex center r . We observe the interval of separations (vortex interior) where $U = \text{const.}$



Alternatively, the mean flow can be characterized by the mean vorticity $\Omega = \partial_r U + U/r$. The mean vorticity behaves $\propto r^{-1}$ in the region. The dependence is saturated at the vortex core where viscosity starts to play an essential role in forming the mean flow. The mean vorticity tends to zero fast outside the vortex.



In the vortex interior fluctuations of the velocity are relatively weak. It is a consequence of the large value of the mean velocity gradient $\sim U/r$, growing toward the center of the vortex. The relative strength of the velocity fluctuations increases as r grows and on the periphery of the vortex the velocity fluctuations become of the order of the mean flow velocity.

The velocity of the mean flow is determined in the reference frame attached to the vortex center. After averaging in time we find a differential rotation characterized by the polar velocity $U(r)$. The stationary equation for it is written as the momentum balance averaged in time

$$\alpha U = -(\partial_r + 2/r)(\Pi - \nu \Sigma).$$

Here Σ is the local shear rate

$$\Sigma = \partial_r U - U/r,$$

of the mean flow and Π is the corresponding component of the Reynolds stress tensor

$$\Pi = \langle v_\varphi v_r \rangle,$$

where v_r and v_φ are the radial and the polar components of the fluctuating velocity.

The small amplitude of the velocity fluctuations implies that the velocity fluctuations inside the vortex are passive. Therefore the fluctuations can be considered in the quasilinear approximation that explains the flat mean velocity profile (Kolokolov, Lebedev 2016,2017). The key goal is to calculate $\langle v_\varphi v_r \rangle$ entering the equation for U .

In the quasilinear approximation we find

$$\partial_t \varpi + (U/r) \partial_\varphi \varpi + (\alpha - \nu \nabla^2) \varpi = \text{curl } f,$$

in the reference frame attached to the vortex center. Even the static pumping (in the laboratory reference frame) has a finite correlation time in the reference frame determined by the statistics of vortex movements.

Solving the equation for ϖ in the shear approximation and averaging the result over the statistics of pumping, one finds

$$\Pi = \langle v_\varphi v_r \rangle = \frac{\epsilon}{\Sigma}, \quad k_f r \gg 1,$$

where α is assumed to be small. Note that the answer is independent of viscosity, though it is zero at $\nu = 0$.

The result can be explained by the energy balance. Assuming space homogeneity due to $k_f r \gg 1$, one finds

$$\Sigma \langle v_1 v_2 \rangle = \epsilon - \nu \langle (\nabla v_\alpha)^2 \rangle - \alpha \langle v^2 \rangle,$$

where $\mathbf{v}$ designates the fluctuating velocity on top of the coherent flow. If the fluctuations are weak then $\Sigma \langle v_1 v_2 \rangle = \epsilon$. It is the energy flux from pumping to the coherent flow.

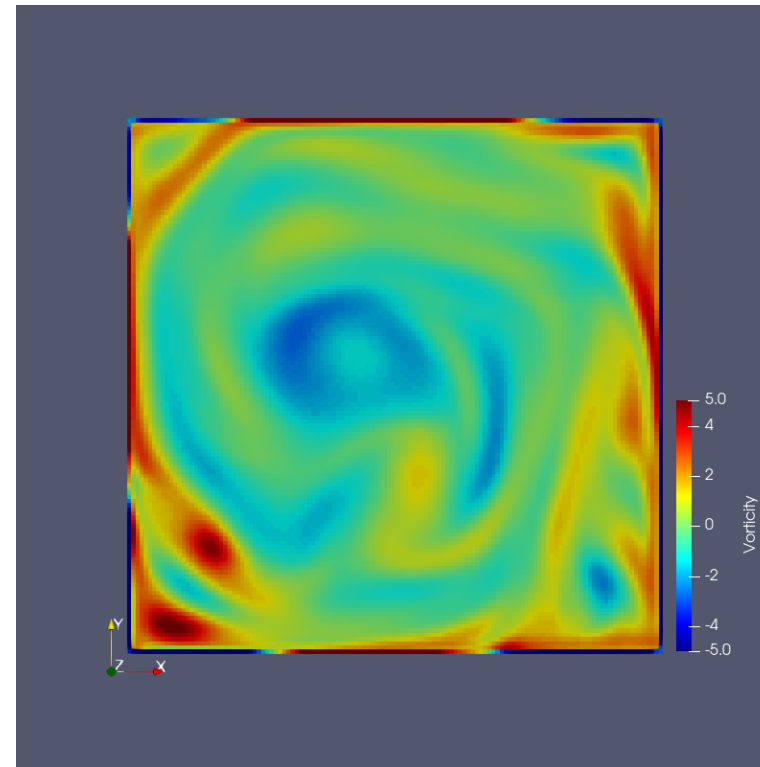
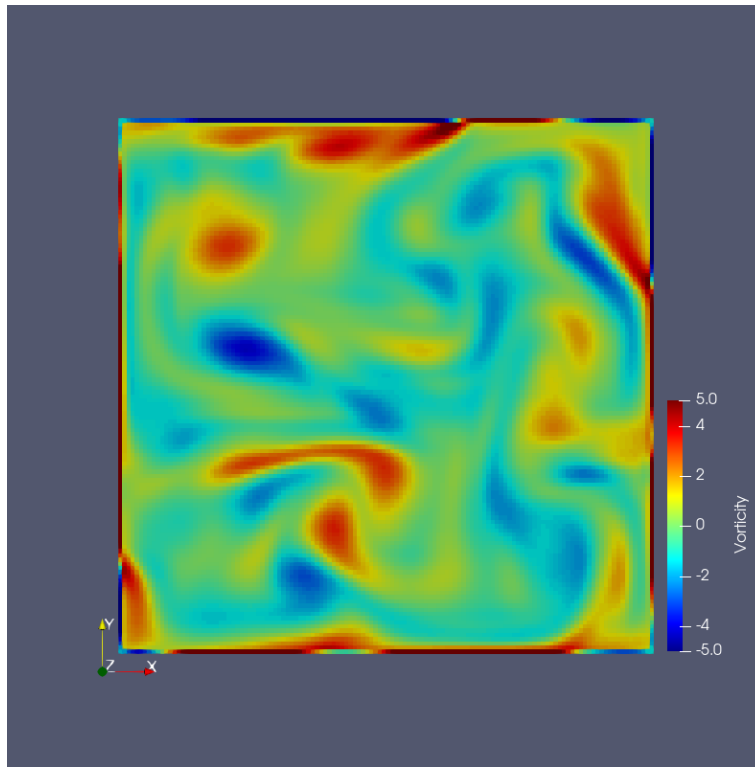
Substituting the expression $\Pi = \epsilon/\Sigma$ into the equation for U and neglecting there the viscous term one obtains

$$\alpha U = -(\partial_r + 2/r)(\epsilon/\Sigma),$$

A solution of the strongly nonlinear equation is the flat profile $U^2 = 3\epsilon/\alpha$. Note that the mean velocity U is of the order of the estimate for large-scale fluctuations.

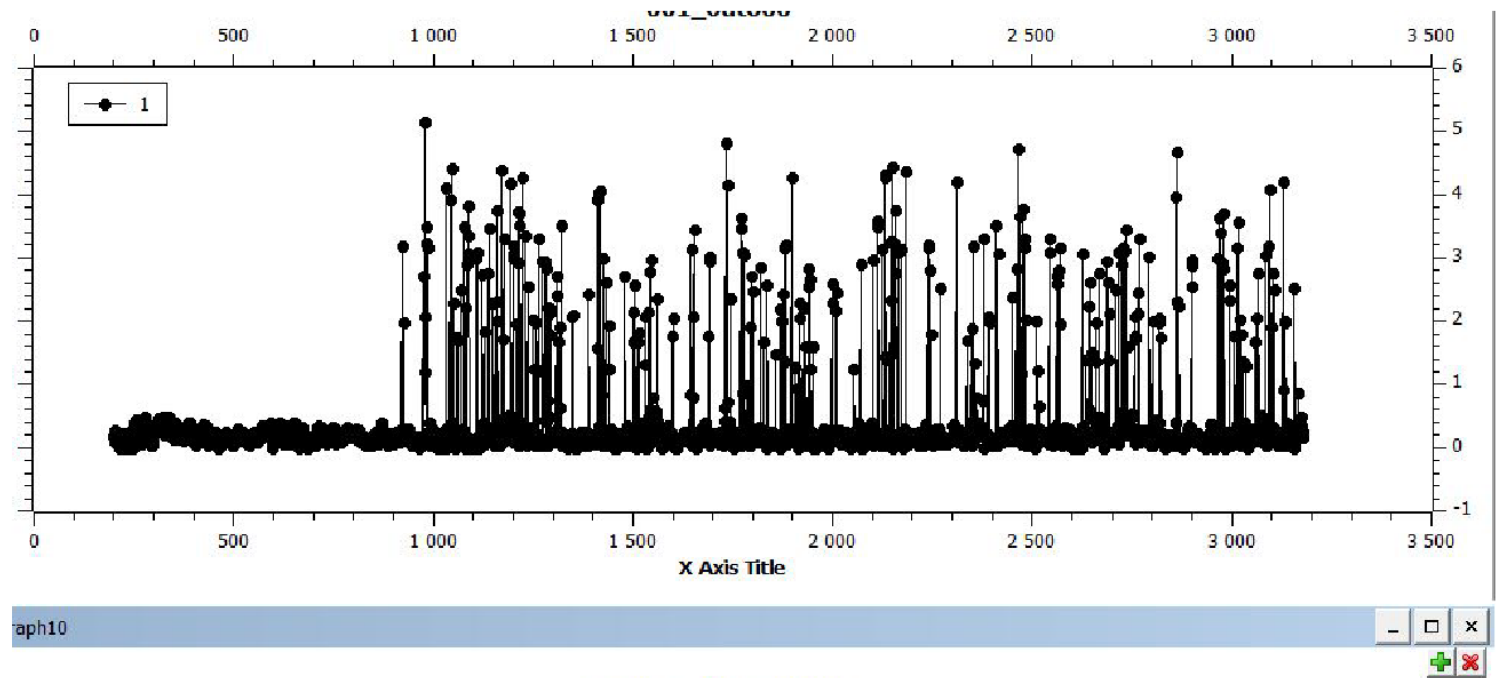
At deriving the expression $\Pi = \epsilon / \Sigma$ in the framework of the quasilinear approximation we neglected α . However, at growing α the average $\Pi = \langle v_\varphi v_r \rangle$ diminishes (Kolokolov, Lebedev, 2020), the characteristic value is $\alpha \sim \nu k_f^2$. For larger α the coherent vortex cannot be supported. What will be the turbulent state in this case?

The answer can be extracted from numerics conducted for the static pumping force in a finite box (Doludenko, Fortova, Kolokolov, Lebedev 2022). At relatively large α we observe large-scale fluctuations with finite correlation time. We present snapshots of the vorticity illustrating the difference in the states.



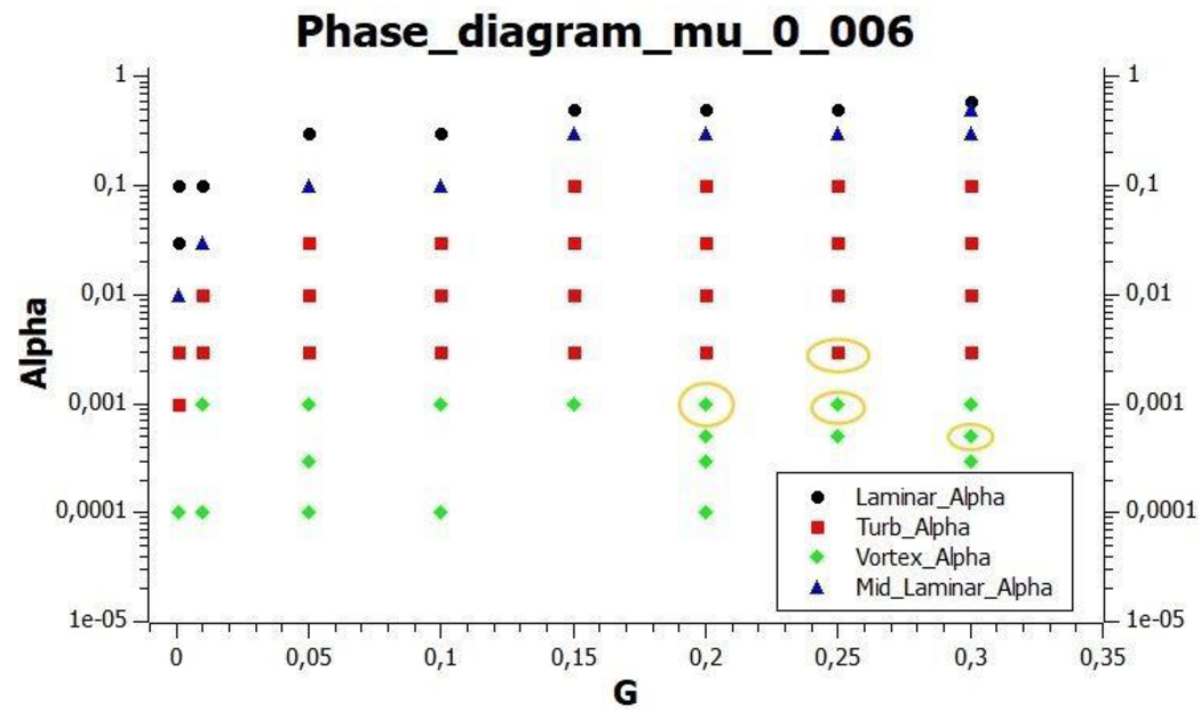
Turbulent state without and with the coherent vortex .

One could find a qualitative criterion to distinguish the states. Let us consider the point of maximal vorticity marked the leading vortex. In the coherent state it moves continuously being the center of the coherent vortex. In the second state the position of the point experiences big jumps at changing the leading vortex. The next picture illustrates the situation.



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Based on the analysis one can draw the “phase diagram” on the plane $G-\alpha$ where G is the amplitude of the pumping force. We see that we pass through the states with coherent vortex, chaotic state without coherent vortex and laminar state as α grows. It is in accordance with the theoretical expectations.



The chaotic state (without coherent vortices) is characterized by strong fluctuations – no small parameter. It is clear that the correlation functions in the state possess no simple scaling. It is the question how to characterize the chaotic state. Its numerical investigation and theoretical analysis are in progress.

Thank you for attention!