



Whitney's problem with random drive

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Abstract

We study stochastic dynamics of an inverted pendulum subject to a horizontal random force (Whitney's problem). The problem admits a unique solution that always remains in the upper half plane. Assuming a white-noise driving, we develop a field-theoretical approach to *statistical description of this non-falling trajectory* (non-FT) based on the supersymmetric formalism of Parisi and Sourlas.

We calculate various correlation functions of the non-FT: single-time distribution function $P(\theta, p)$ of the angle and its velocity, multi-time correlators, and Lyapunov exponent.

Whitney's problem

Inverted pendulum with the horizontal driving $f(t)$:

$$\ddot{\theta} = \omega^2 \sin \theta + f(t) \cos \theta. \quad (1)$$

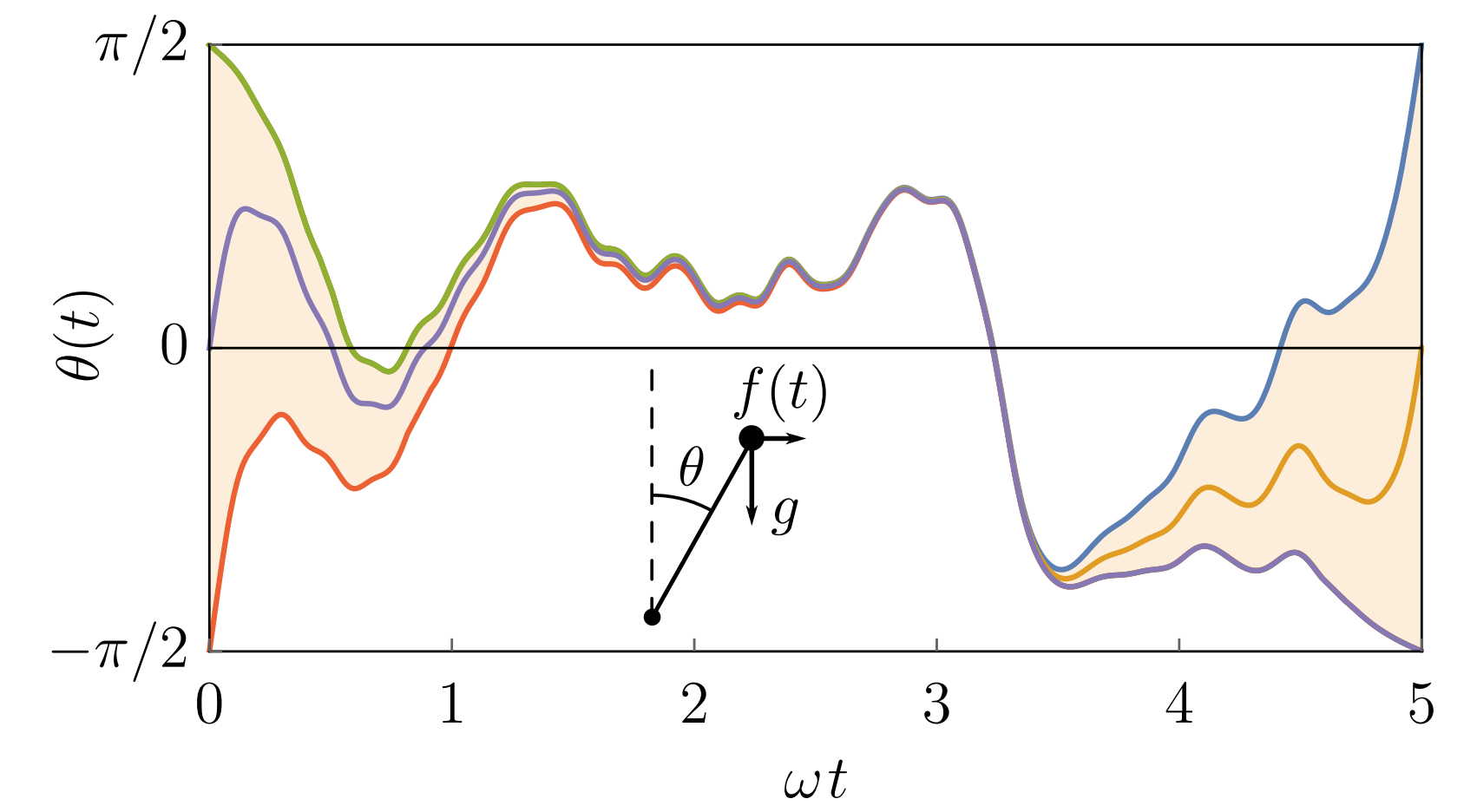
- **Non-falling trajectories (non-FT)** are such solutions of Eq. (1), which satisfy $-\pi/2 < \theta(t) < \pi/2$ for all times $t \in (0, T)$.
- **Existence:** For any time interval $(0, T)$, there exists a non-FT with given boundary conditions $\theta(0)$ and $\theta(T)$ in the strip, provided $f(t)$ is smooth enough [1].
- **Uniqueness:** Such a non-FT is unique [2].
- In the limit $T \rightarrow \infty$, a non-FT turns into a unique **never-falling trajectory (NFT)**.

Stochastic Whitney's problem

We analyze statistics of the non-FT assuming

$$\langle f(t)f(t') \rangle = 2\alpha\delta(t-t').$$

The only dimensionless parameter: α/ω^3 .



Parisi-Sourlas Method

Generating functional (number of solutions) [3]:

$$Z = \int \mathcal{D}\theta \delta(-\theta'' + F(\theta)) \left| \det(-\partial^2 + F'(\theta)) \right|,$$

where $F(\theta, t) = \omega^2 \sin \theta + f(t) \cos \theta$.

- Replace $|\det(\dots)|$ by $\det(\dots)$.
- Rewrite $\det(\dots)$ using Grassman fields $\chi(t)$ and $\bar{\chi}(t)$.
- Express $\delta(\dots)$ via integration over a new field $\lambda(t)$.
- Gaussian averaging over $f(t)$.

$$\langle Z \rangle = \int \mathcal{D}\theta \mathcal{D}\lambda \mathcal{D}\bar{\chi} \mathcal{D}\chi \exp \left[\int dt \left(i\dot{\lambda}\dot{\theta} + \bar{\chi}\dot{\chi} + \mathcal{V} \right) \right]$$

$$\mathcal{V} = \omega^2 (i\lambda \sin \theta + \bar{\chi}\chi \cos \theta) + \alpha (i\lambda \cos \theta - \bar{\chi}\chi \sin \theta)^2$$

Fourier transform from λ to $p = \dot{\theta}$:

$$\Psi(\theta, p) = \int \Psi(\theta, \lambda) e^{-ip\lambda} (d\lambda)$$

It is the uniqueness of the non-FT that justifies dropping the absolute value of $\det(\dots)$, making the Parisi-Souras technique appropriate for Whitney's problem.

Transfer matrix method

Path integral for $\langle Z \rangle$ can be reduced [4] to the imaginary-time Schrödinger equation for the wave function $\Psi + \bar{\chi}\chi\Phi$:

$$\frac{\partial}{\partial t} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} = -\mathcal{H} \begin{pmatrix} \Psi \\ \Phi \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} L & -1 \\ V_2 & L \end{pmatrix}, \quad (2)$$

L is the Fokker-Planck operator for the Kramers problem:

$$L = p\partial_\theta + \omega^2 \sin \theta \partial_p - \alpha \cos^2 \theta \partial_p^2, \quad (3)$$

$$V_2 = -\omega^2 \cos \theta - \alpha \sin 2\theta \partial_p. \quad (4)$$

Boundary conditions:

$$\Psi(\theta = \pi/2, p < 0) = \Psi(-\pi/2, p > 0) = 0. \quad (5)$$

Scalar product:

$$\langle \Psi_L | \Psi_R \rangle = \int dp d\theta \hat{\Psi}_L^T(\theta, -p) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \hat{\Psi}_R(\theta, p). \quad (6)$$

BRST symmetry

$\mathcal{H}$ -evolution respects the symmetry $\partial_\theta \Psi = -\partial_p \Phi$. Hence, both components can be reduced to a single function:

$$\Psi(\theta, p) = \partial_p \psi(\theta, p); \quad \Phi(\theta, p) = -\partial_\theta \psi(\theta, p). \quad (7)$$

Evolution equation for ψ :

$$\partial_t \psi = L\psi. \quad (8)$$

Initial and final conditions

For trajectories with a given initial (final) angle θ_0 or momentum p_0 , the wave function $\hat{\Psi}$ has the form:

$$\hat{\Psi}_{\theta_0} = \begin{pmatrix} 0 \\ \delta(\theta - \theta_0) \end{pmatrix}; \quad \hat{\Psi}_{p_0} = \begin{pmatrix} \delta(p - p_0) \\ 0 \end{pmatrix}. \quad (9)$$

Different boundary problems on a finite time interval:

- $Z(\theta_i \rightarrow \theta_f) = \langle \hat{\Psi}_{\theta_f} | e^{-t\mathcal{H}} | \hat{\Psi}_{\theta_i} \rangle = 1$
- $Z(\theta_i \rightarrow p_f) = \langle \hat{\Psi}_{p_f} | e^{-t\mathcal{H}} | \hat{\Psi}_{\theta_i} \rangle \neq 1$
- $Z(p_i \rightarrow p_f) = \langle \hat{\Psi}_{p_f} | e^{-t\mathcal{H}} | \hat{\Psi}_{p_i} \rangle \neq 1$

Average and correlators

A physical operator $O(\theta, p)$ has the matrix representation:

$$\hat{O} = \begin{pmatrix} O(\theta, p) & 0 \\ 0 & 0 \end{pmatrix} \quad (10)$$

Correlation functions to be calculated as

$$\langle O_1(t_1) O_2(t_2) \dots \rangle = \langle \hat{\Psi}_f | \dots \hat{O}_2 e^{-t\mathcal{H}(t_2-t_1)} \hat{O}_1 e^{-\mathcal{H}(t_1-t_i)} | \hat{\Psi}_i \rangle$$

The PDF corresponds to $O(t) = \delta(\theta(t) - \theta)\delta(p(t) - p)$:

$$P(\theta, p) = \Psi_L(\theta, -p, t) \Phi_R(\theta, p, t) + \Phi_L(\theta, -p, t) \Psi_R(\theta, p, t)$$

Physical meaning of Ψ , Φ and ψ

$\Psi(\Phi)$ is expressed via the conditional probability distribution of p at a given θ (θ at a given p):

$$\Psi(\theta, p) = P(p|\theta, t), \\ \Phi(\theta, p) = P(\theta|p, t).$$

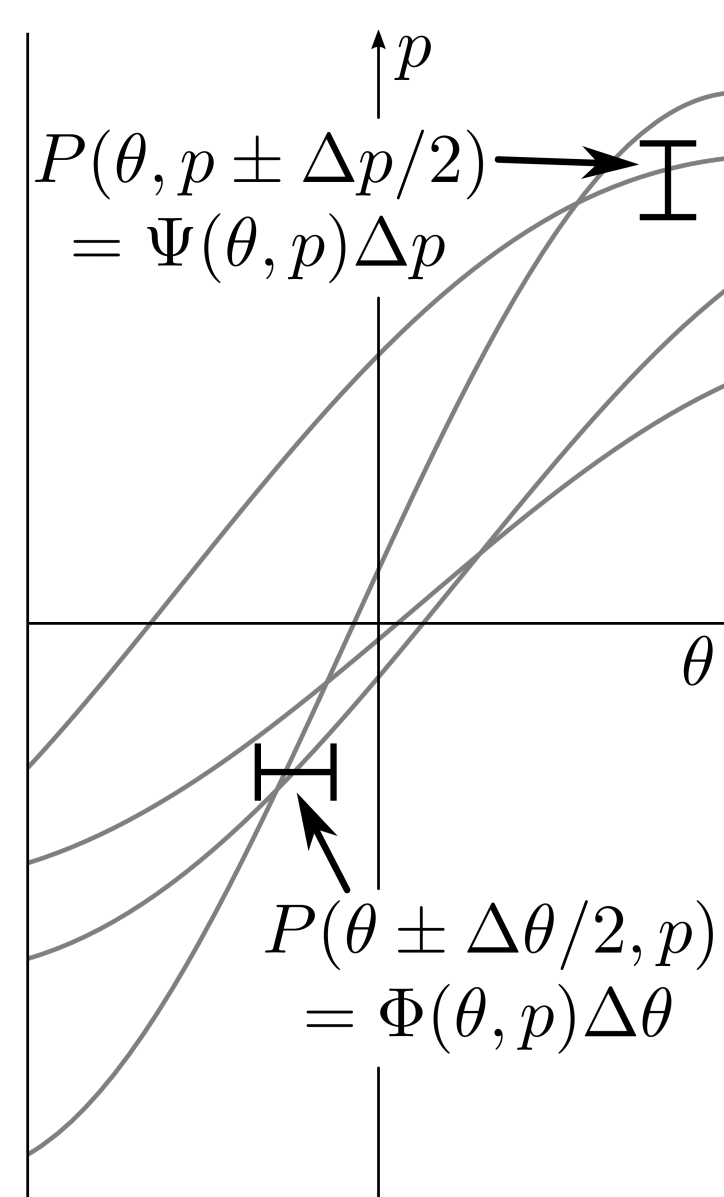
The ensemble of various characteristics of Eq. (1) is described by the 1D Burgers equation:

$$\partial_t p + p\partial_\theta p = F(\theta, t).$$

The function ψ is related to the momentum *profile* $p(\theta, t)$:

$$\psi(\theta, p, t) = \Theta(p(\theta, t) - p),$$

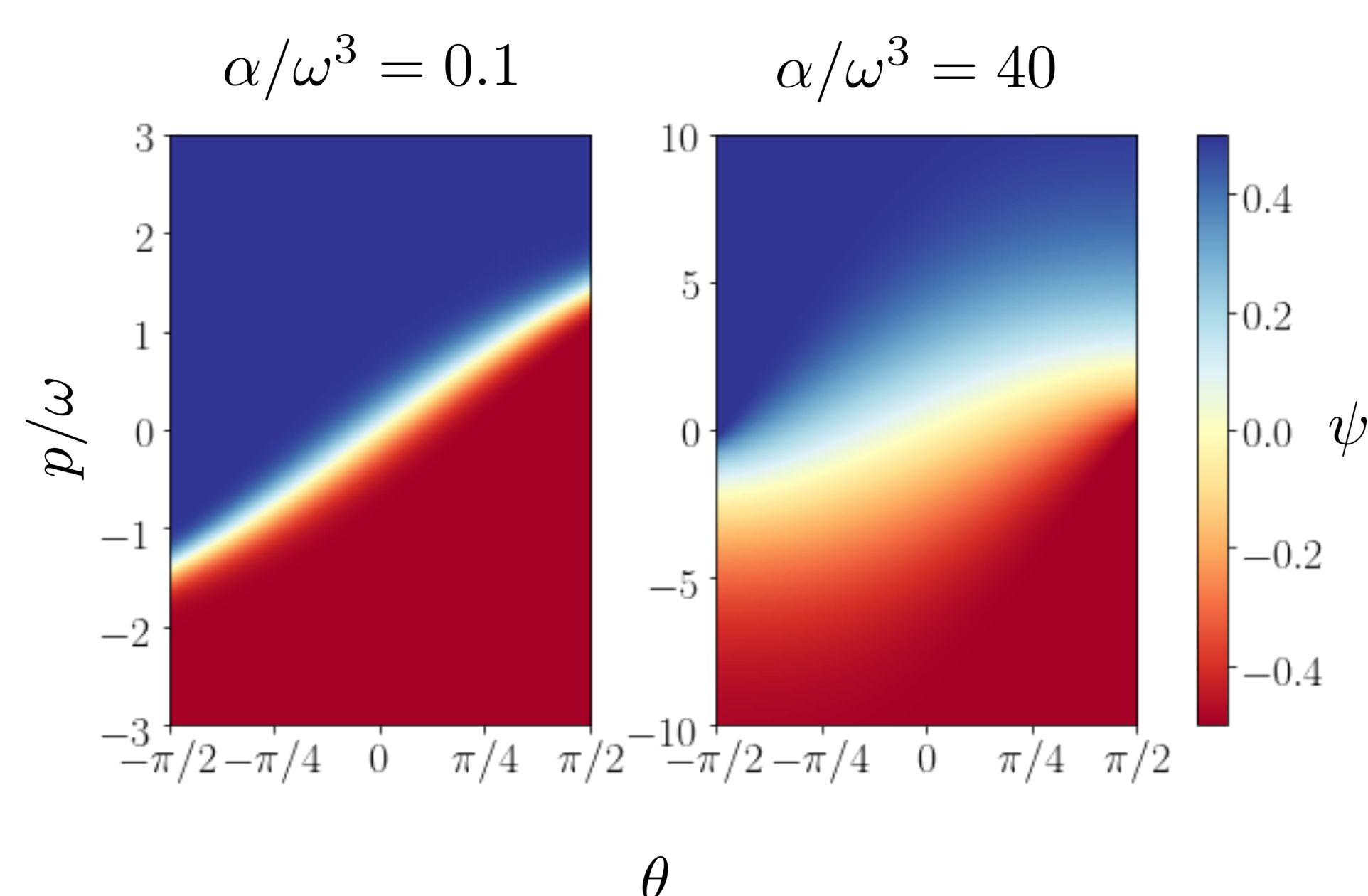
where $\Theta(x)$ is the Heaviside step function.



Zero mode for the NFT case

Numerical solution for the zero mode $\psi(p, \theta)$ of equation:

$$L\psi = 0 \quad (11)$$



Correlators for weak noise ($\alpha^3/\omega \ll 1$)

- One-point correlator for NFT: $\langle \theta^2 \rangle = \alpha/2\omega^3$.
- Two-point correlation function for NFT: $\langle \theta(0)\theta(t) \rangle = \langle \theta^2 \rangle (1 + \omega t) e^{-\omega t}$
- Boundary problem with $\theta(0)$, the one-point correlator: $\langle \theta(t) \rangle = \theta(0) e^{-\omega t}$.

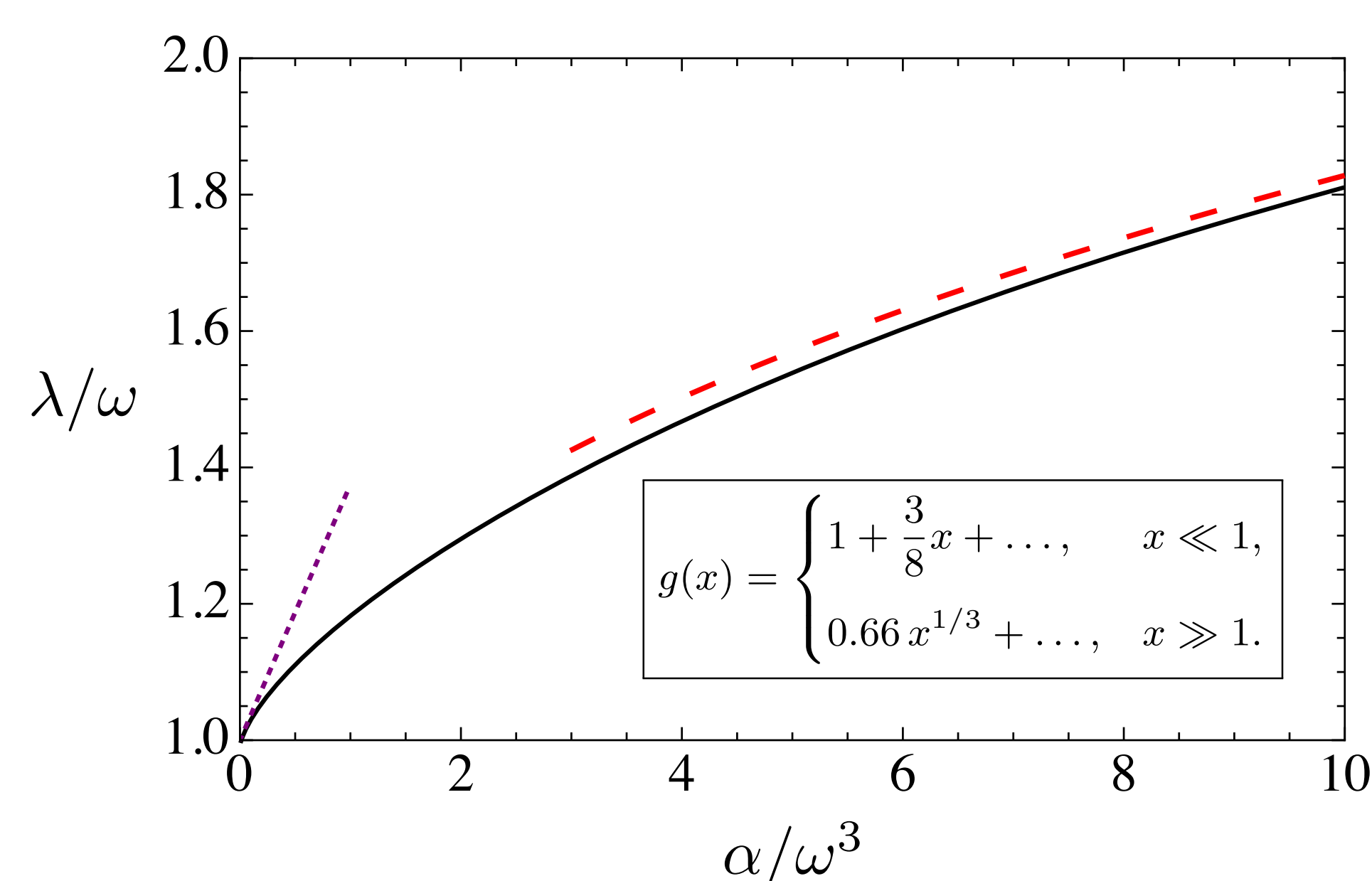
Lyapunov exponent

The Lyapunov exponent:

$$\lambda = -\lim_{t \rightarrow \infty} \frac{\partial \ln \langle \theta(0)\theta(t) \rangle}{\partial t} = -\lim_{t \rightarrow \infty} \frac{\partial \ln \langle \theta(t) \rangle_{\theta_0}}{\partial t}. \quad (14)$$

General expression for λ :

$$\lambda = \omega g(\alpha/\omega^3). \quad (15)$$

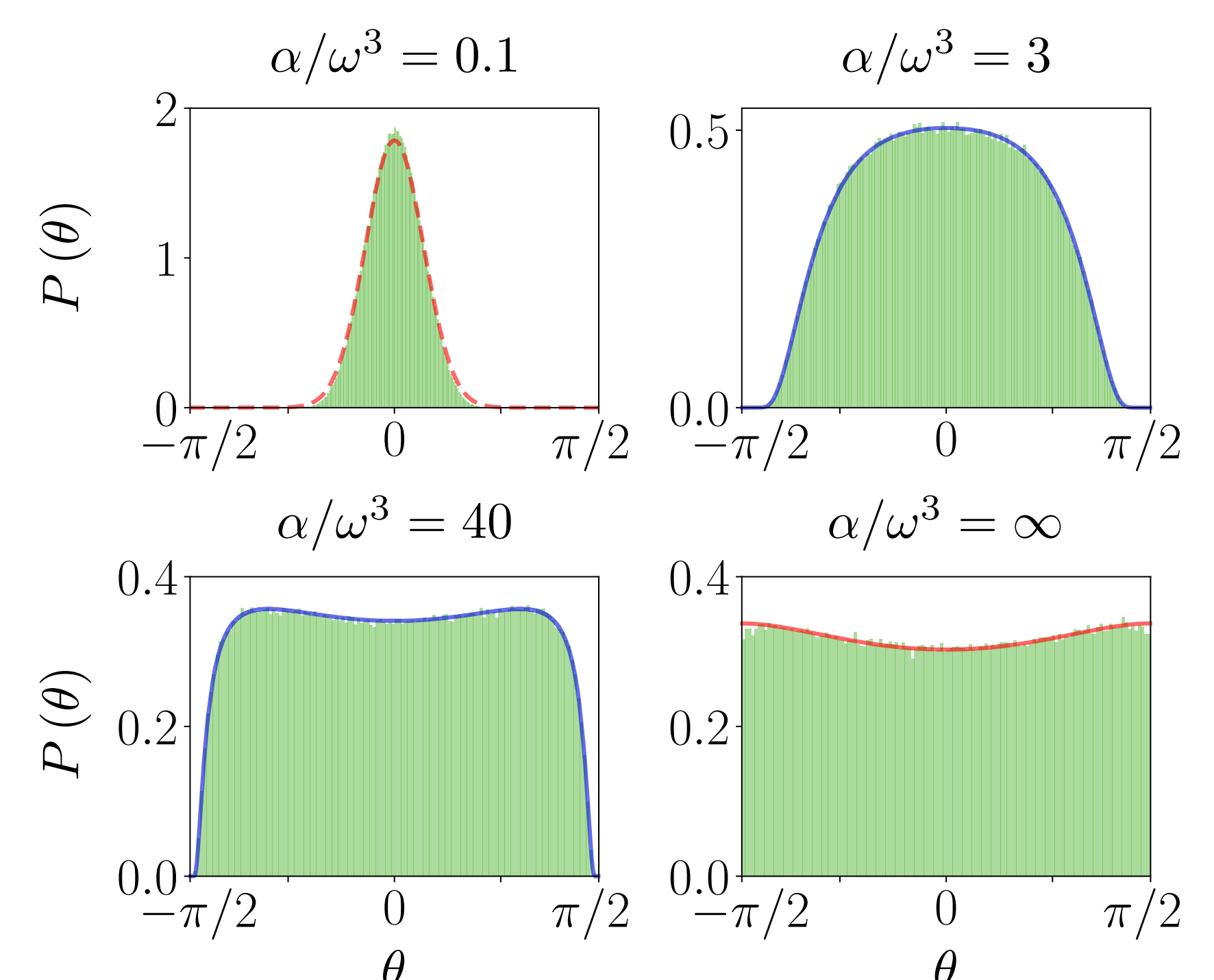


Details in the talk by M. Skvortsov

PDF for NFT

Probability distribution function of the NFT angle, $P(\theta)$.

- Green histograms are obtained by Monte-Carlo simulation.
- Numeric solutions of (11) are shown by blue lines.
- Solid red line is the analytical solution for $\omega = 0$.



References

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- [5] N. A. Stepanov and M. A. Skvortsov, JETP Letters, **112**, 376 (2020).